# Non-Archimedean Linear Operators and Applications

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*In memory of Mabaye Galledou,*
*my grandmother*

# Contents

# Preface

This self-contained book provides the reader with a comprehensive presentation of the author's recent investigations on operator theory over non-Archimedean Banach and Hilbert spaces. It therefore completes topics uncovered in the author's recent book, "*An Introduction to Classical and p-adic Operator Theory and Applications*", which has recently been published by *Nova Science Publishers*. Moreover, it offers several new research lines, which are, in most part, inspired by their classical counterparts.

*Chapter 1* is devoted to basic tools related to (complete) non-Archimedean valued fields needed in the sequel. Among other things, the ideas of construction of the fields of $p$-adic numbers and that of formal Laurent series are discussed. Many key results on the topology of the field of $p$-adic numbers are also discussed.

A valued field $(\mathbb{K}, |\cdot|)$ is called non-Archimedean whether the following stronger (triangle) inequality holds

$$|\sigma + \varsigma| \leq \max(|\sigma|, |\varsigma|), \quad \text{for all } \sigma, \varsigma \in \mathbb{K}$$

with equality whenever $|\sigma| \neq |\varsigma|$.

The non-Archimedean operator theory consists of studying the analogues of the classical operator theory when the (complete) non-Archimedean field $(\mathbb{K}, |\cdot|)$ plays the role usually played by the fields of real or complex numbers. Classical examples of such fields include $\mathbb{Q}_p$ the field of $p$-adic numbers ($p \geq 2$ prime), $\mathbb{C}_p$ the field of $p$-adic complex numbers, and the field of formal Laurent series.

*Chapter 2* introduces basic properties of non-Archimedean Banach spaces, free Banach spaces, and non-Archimedean Hilbert spaces needed in the sequel. One should point out that this is a key chapter, especially for beginners.

*Chapter 3* is entirely devoted to bounded linear operators on non-Archimedean Banach and Hilbert spaces. In particular, one takes a closer look into the existence of an adjoint for a given bounded linear operator with respect to the non-Archimedean "inner product". Examples of bounded linear operators, which do not have adjoint also are discussed. New developments on perturbations of non-Archimedean bases are also investigated. A nontrivial example of an orthogonal base constructed by perturbation of the canonical orthogonal base of the non-Archimedean Hilbert space $\mathbb{E}_\omega$, is also given. In the middle of this chapter, special attention is paid to Hilbert-Schmidt and completely continuous operators. Some of the results go along the classical line and others deviate from it. For the most part, their classical counterparts inspire the statements of the results we present. However, their proofs may depend heavily on the non-Archimedean nature of $\mathbb{E}_\omega$ and the ground field $\mathbb{K}$. In contrast with the classical context, *the definition of a Hilbert-Schmidt operator does depend on the base*. Furthermore, the adjoint of Hilbert-Schmidt operator does exists and also is a Hilbert-Schmidt operator, and each Hilbert-Schmidt operator is completely continuous. As in the classical setting, a natural inner product is considered for Hilbert-Schmidt operators. To illustrate abstract results, several examples are discussed at the end of this chapter.

*Chapter 4* examines non-Archimedean unbounded linear operators recently introduced in the literature in Diagana **[17]**. Several examples are discussed including those arising on the free Banach space $C(\mathbb{Z}_p, \mathbb{Q}_p)$ of all continuous functions from the ring of integers, $\mathbb{Z}_p$, into $\mathbb{Q}_p$, the field of $p$-adic numbers.

*Chapter 5* studies bounded and unbounded non-Archimedean bilinear forms. The closure and the closedness of (unbounded) non-Archimedean bilinear forms are investigated. In particular, necessary conditions for the closedness of the form sum of closed non-Archimedean bilinear forms are given.

*Chapter 6* consists of a preliminary work of the author on functions and fractional powers of linear operators on non-Archimedean Hilbert spaces. This chapter is illustrated by a few examples including functions of a specific (symmetric) square matrix over $\mathbb{Q}_p \times \mathbb{Q}_p$.

*Chapter 7* provides the reader with a brief and recent conceptualization of semigroups within the non-Archimedean framework due to Diagana **[23]**. Non-Archimedean semigroups play an important role in the solvability of $p$-adic differential and partial differential equations as strong (mild) solutions to the Cauchy problem related to several classes of differential and partial differential equations in the classical setting can be expressed through $C_0$-semigroups. An example of solvability of differential equations using the concept of a non-Archimedean semigroup is discussed at the end of this chapter.

*"Non-Archimedean Linear Operators and Applications"* is aimed at expert readers, young researchers, beginning graduate and advanced undergraduate students, who are interested in non-Archimedean functional analysis and operator theory as well as their wide range of applications. The basic background for the understanding of the material presented is timely provided throughout the text. Furthermore, the book offers several kinds of examples, ranging from routine to challenging. The end of each chapter offers some useful bibliographical notes and open problems.

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Washington, D.C. Toka Diagana

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*Chapter 1*

# Non-Archimedean Valued Fields

## 1.1 Introduction

This chapter provides the reader with the basic tools on the so-called *Non-Archimedean* valued fields. Examples of those fields include, but not limited to, $\mathbb{Q}_p$ ($p \geq 2$ is prime), the field of *p-adic numbers*, $\mathbb{C}_p$, the field of *complex p-adic numbers*, and $\mathbb{K}((x))$, the field of *formal Laurent series*. All these fields satisfy the so-called non-Archimedean (or ultrametric) triangle inequality formally given by

$$|\sigma + \varsigma| \leq \max(|\sigma|, |\varsigma|), \quad \text{for all } \sigma, \varsigma \in \mathbb{K}$$

with equality whenever $|\sigma| \neq |\varsigma|$, where $|\cdot|$ is the corresponding absolute value.

A valued field, which does not satisfy the latter inequality is called Archimedean. Examples of Archimedean fields include $(\mathbb{R}, |\cdot|)$, the field of real numbers equipped with its natural absolute value, $(\mathbb{C}, |\cdot|)$, the field of complex numbers equipped with its absolute value, and many others. Using the non-Archimedean triangle inequality, one derives several properties of those non-Archimedean fields. In term of geometry, we will see that in a non-Archimedean field, each ball is both closed and open, the parallelogram law could turn into an inequality[1], every triangle is isosceles, etc.

One of the main objectives of this book is to make the (non-Archimedean) material utilized accessible to many people including those who are not familiar with number theory. For that, proofs of the results, which require some advanced algebraic concepts will simply be omitted.

This chapter is organized as follows: Section 1.2 examines topologies of non-Archimedean valued fields. Section 1.3 discusses the construction of $\mathbb{Q}_p$, the field of $p$-adic numbers. Section 1.4 introduces $\mathbb{K}((x))$, the so-called field of formal Laurent series.

Throughout the rest of the book $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{C}_p$, $\mathbb{Z}_p$, $\mathbb{N}$, and $\mathbb{Z}$ stand for the fields of real, complex, rational, $p$-adic, $p$-adic complex numbers, the ring of $p$-adic integers, the set of natural integers, and the set of all integers, respectively.

[1] In a non-Archimedean valued field $(\mathbb{K}, |\cdot|)$, the non-Archimedean parallelogram law states that for all $x, y \in \mathbb{K}$, $|x+y|^2 + |x-y|^2 \leq 2\max(|x|^2, |y|^2)$.

## 1.2 Non-Archimedean Valued Fields

This section introduces the so-called non-Archimedean valued fields. Those fields will play a key role throughout the rest of the book. We will especially emphasis on their topological properties, which are mainly driven by the non-Archimedean (or ultrametric) inequality [2]. In most of our examples, we will make extensive use of the field of the $p$-adic numbers, known as the simplest (complete) non-Archimedean valued field. The literature on the algebraic properties of those non-Archimedean fields is very extensive; here, we omit those details and refer the reader to most good books in non-Archimedean functional analysis, especially Amice**[2]**, Schikhof**[64]**, Gouvêa**[33]**, Koblitz**[42]**, Mahler**[50]**, Monna**[51]**, van Rooij**[62]**, and many others.

### 1.2.1 Basic Definitions

**Definition 1.** A field $\mathbb{K}$ is said to be *non-Archimedean* if it is endowed with an absolute value $|\cdot| : \mathbb{K} \mapsto [0,\infty)$, which satisfies the following conditions:

(1) $|x| = 0$ if and only if $x = 0$;
(2) $|xy| = |x| . |y|$, for all $x, y \in \mathbb{K}$; and
(3) $|x+y| \leq \max(|x|,|y|)$ for all $x, y \in \mathbb{K}$.

As it had previously been mentioned, the property [Definition 1.3.1, (3)], is called the non-Archimedean triangle inequality. An immediate consequence of the non-Archimedean triangle inequality is the following:

(4) $|x+y| = \max(|x|,|y|)$ whenever $|x| \neq |y|$, where $x, y \in \mathbb{K}$.

As in the classical setting, if $(\mathbb{K}, |\cdot|)$ is a non-Archimedean field, then its absolute value $|\cdot|$ induces a metric $d$ on $\mathbb{K} \times \mathbb{K}$ defined by

$$d(u,v) = |u-v|, \quad u,v \in \mathbb{K}.$$

The metric $d$ enables to consider the notion of convergence in $\mathbb{K}$. Basically, one says that the sequence $(u_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ converges to $u \in \mathbb{K}$ whether

$$d(u_t,u) = |u_t - u| \to 0 \text{ as } t \to \infty.$$

Now if the metric space $(\mathbb{K}, d)$ is *complete*[3], one says that $(\mathbb{K}, |\cdot|)$ is a complete non-Archimedean field. Otherwise, one can always complete it and denote its completion by $(\tilde{\mathbb{K}}, |\cdot|)$. It can be easily shown that $(\tilde{\mathbb{K}}, |\cdot|)$ is also a non-Archimedean field, which obviously is complete.

In contract with the classical setting, in a non-Archimedean field, every triangle is isosceles. This obviously is a consequence of the non-Archimedean triangle inequality.

[2] Throughout the book, the word "non-Archimedean" will be preferred to that of "ultrametric".

[3] A metric space $(\mathbb{K}, d)$ is said to be complete if every cauchy sequence of elements in $\mathbb{K}$ converges. Note that in the non-Archimedean setting, a sequence $(u_n)_{n\in\mathbb{N}} \subset (\mathbb{K}, d)$ is a Cauchy sequence if and only if $d(u_{n+1}, u_n) \to 0$ as $n \to \infty$, see Proposition2.

**Proposition 1.** *Let* $(\mathbb{K}, |\cdot|)$ *be (metric space) a non-Archimedean field and let* $x, y, z \in \mathbb{K}$ *such that* $|x-z| \neq |z-y|$. *Then*

$$|x-y| = \max(|x-z|, |z-y|).$$

*Proof.* Suppose that $|x-z| < |z-y|$. (The proof for the case $|z-y| < |x-z|$ follows along the same line.) It follows that

$$|x-y| \leq \max(|x-z|, |z-y|) = |z-y|.$$

Suppose that $|x-y| < |z-y|$. Consequently,

$$|z-y| \leq \max(|z-x|, |x-y|) < |z-y|,$$

which obviously is in contradiction with the fact $|x-y| < |z-y|$, by assumption, and therefore,

$$|x-y| = \max(|x-z|, |z-y|) = |z-y|.$$

**Definition 2.** Let $(\mathbb{K}, |\cdot|)$ be a non-Archimedean field. A sequence $(x_t)_{t\in\mathbb{N}}$ of $\mathbb{K}$ is said to converge to $x$ if for each $\varepsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $|x_t - x| < \varepsilon$ whenever $t \geq t_0$. A sequence $(x_t)_{t\in\mathbb{N}}$ of $\mathbb{K}$ is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $|x_t - x_s| < \varepsilon$ whenever $s \geq t \geq t_0$.

In contrast with the classical, the following useful result, which also is a consequence of the non-Archimedean inequality, holds:

**Proposition 2.** *Let* $(\mathbb{K}, |\cdot|)$ *be a non-Archimedean field. A sequence* $(x_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ *is a Cauchy sequence if and only if*

$$|x_{t+1} - x_t| \to 0 \quad as \quad t \to \infty.$$

*Proof.* Let $(x_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a Cauchy sequence. Therefore, for each $\varepsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $|x_t - x_s| < \varepsilon$ whenever $s \geq t \geq t_0$. In particular, $|x_{t+1} - x_t| \to 0$ whenever $t \to \infty$.

Conversely, suppose that $|x_{t+1} - x_t| \to 0$ as $t \to \infty$ and write,

$$x_s - x_t = x_s - x_{s-1} + x_{s-1} - \ldots - x_{t+1} + x_{t+1} - x_t.$$

Combining all terms of the form $x_r - x_{r-1}$ for $r = s, \ldots, t$ and using [Definition 1.3.1, (3)] it easily follows that $|x_s - x_t| \to 0$ as $t, s \to \infty$, and hence $(x_t)_{t\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$.

In contrast with Proposition 2 above, in the next result, the completion of the field $(\mathbb{K}, |\cdot|, d)$ is required.

**Proposition 3.** *Let* $(\mathbb{K}, |\cdot|, d)$ *be a complete non-Archimedean field. The series* $\sum_{t=0}^{\infty} x_t$ *converges to some* $S \in \mathbb{K}$ *if and only if* $x_t \to 0$ *as* $t \to \infty$.

*Proof.* Let $S_t = \sum_{r=0}^{t} x_r$. It is then clear that $x_t = S_t - S_{t-1}$. Thus, if $S_t \to S$, then $x_t \to 0$ in $\mathbb{K}$, as $t \to \infty$.

Conversely, if $x_t \to 0$, then $(S_t)_{t\in\mathbb{N}}$ is a Cauchy sequence, by Proposition 2. Now since $\mathbb{K}$ is complete, there exists $S \in \mathbb{K}$ such that $S_t \to S$ as $t \to \infty$.

Let $(\mathbb{K}, |\cdot|)$ be a non-Archimedean valued field. Define the balls $\Omega(x,r)$, $\Omega'(x,r)$, and the sphere $S(x,r)$ of $\mathbb{K}$ by:

$$\Omega(x,r) := \{y \in \mathbb{K} : |x-y| \le r\},$$

$$\Omega'(x,r) := \{y \in \mathbb{K} : |x-y| < r\}, \text{ and}$$

$$S(x,r) := \{y \in \mathbb{K} : |x-y| = r\}.$$

In particular, we set $\Omega_r = \Omega(0,r)$, $\Omega'_r = \Omega(0,r)$, and $S_r = S(0,r)$.

**Proposition 4.** *Let $x \in \mathbb{K}$ and let $r > 0$ be a real number. If $z \in \Omega(x,r)$, then $\Omega(z,r) = \Omega(x,r)$.*

*Proof.* Let $y \in \Omega(z,r)$, that is, $|z-y| \le r$. Obviously,

$$|y-x| = |(y-z)+z-x| \le \max(|y-z|, |z-x|) \le r,$$

and hence $y \in \Omega(x,r)$, or $\Omega(z,r) \subset \Omega(x,r)$. Now, from the fact that $z \in \Omega(x,r)$ it is clear that $x \in B(z,r)$.

Arguing as previously, it follows that $\Omega(x,r) \subset \Omega(z,r)$. From both inclusions it follows that $\Omega(x,r) = \Omega(z,r)$.

From Proposition 4, we obtain the following result:

**Theorem 1.** *Let $x \in \mathbb{K}$ and let $r > 0$ be a real number. The balls $\Omega(x,r)$, $\Omega'(x,r)$ and the sphere $S(x,r)$ are respectively closed and open in $\mathbb{K}$.*

Note that subsets of $\mathbb{K}$, which are both closed and open are commonly called *clopens.*

In addition to the above, the unit balls $\Omega_1$ and $\Omega'_1$ satisfy the following properties:

**Proposition 5.** *If $(\mathbb{K}, |\cdot|)$ is a non-Archimedean field, then*

*(1) $\Omega_1$ is a subring of $\mathbb{K}$ with identity $1_{\mathbb{K}}$;*
*(2) $\mathbb{K}$ is the field of fractions of $\Omega_1$;*
*(3) $\Omega'_1$ is an ideal of $\Omega_1$;*
*(4) $\Omega'_1$ is the unique maximal ideal of $\Omega_1$; and*
*(5) The quotient $\Omega_1/\Omega'_1$ is a field.*

The proof of Proposition 5 can be found in Gouvêa **[33]**. Note that the balls $\Omega_1$, $\Omega'_1$, and the quotient $\Omega_1/\Omega'_1$ are respectively called the valuation ring of $(\mathbb{K}, |\cdot|)$, the valuation ideal of $(\mathbb{K}, |\cdot|)$, and the residue field of $\mathbb{K}$ with respect to $\Omega'_1$.

### 1.2.2 The $t$-Vector Space $\mathbb{K}^t$

The $(\mathbb{K},|\cdot|)$ be a non-Archimedean field. As for the n-dimensional real and complex numbers $\mathbb{R}^t$ and $\mathbb{C}^t$, one defines $\mathbb{K}^t$ as follows:

$$\mathbb{K}^t := \{x = (x_1, x_2, x_3, ...., x_t) : x_r \in \mathbb{K}, \text{ for } r = 1, ..., t\}.$$

It is clear that $\mathbb{K}^t$ is a vector space over $\mathbb{K}$. Moreover, one equips it with the absolute value

$$|x|_t = \max_{1\leq r\leq t} |x_r|, \quad \forall x = (x_1, x_2, x_3, ...., x_t) \in \mathbb{K}^t.$$

Clearly, for all $x = (x_1, x_2, x_3, ...., x_t)$, $y = (y_1, y_2, y_3, ...., y_t) \in \mathbb{K}^t$,

$$\begin{aligned} |x+y|_t &= \max_{1\leq r\leq t} |x_r + y_r| \\ &\leq \max_{1\leq r\leq t} \max(|x_r|, |y_r|) \\ &= \max(\max_{1\leq r\leq t} |x_r|, \max_{1\leq r\leq t} |y_r|) \\ &= \max(|x|_t, |y|_t), \end{aligned}$$

and hence $|\cdot|_t$ is a non-Archimedean absolute value over $\mathbb{K}^t$.

In view of the above, it is easy to check that the following *parallelogram* law holds: for all $x, y \in \mathbb{K}^t$,

$$|x+y|_t^2 + |x-y|_t^2 \leq 2\max(|x|_t^2, |y|_t^2)$$

with equality whenever $|x|_t \neq |y|_t$.

It is not hard to see that many algebraic and topological properties of $\mathbb{K}$ can carried over to $\mathbb{K}^t$. In particular, if $(\mathbb{K},|\cdot|)$ is complete, then so is $(\mathbb{K}^t, |\cdot|_t)$.

Let $\Omega(x,r)$ and $S(x,r)$ denote respectively the ball and the sphere of $\mathbb{K}^t$ centered at $x \in \mathbb{K}^t$ with radius $r > 0$:

$$\Omega(x,r) := \{y \in \mathbb{K}^t : |x-y|_t \leq r\}, \text{ and}$$

$$S(x,r) := \{y \in \mathbb{K}^t : |x-y|_t = r\}.$$

Clearly, $\Omega(x,r)$ and $S(x,r)$ are respectively clopens. Moreover, one can easily check that if $x = (x_1, x_2, x_3, ...., x_t) \in \mathbb{K}^t$, then

$$\Omega(x,r) = \Omega(x_1,r) \times \Omega(x_2,r) \times \Omega(x_3,r) \times ... \times \Omega(x_t,r).$$

In addition to the above, an "inner product" is defined on $\mathbb{K}^t$ as follows: for all $x = (x_1, x_2, x_3, ...., x_t)$, $y = (y_1, y_2, y_3, ...., y_t) \in \mathbb{K}^t$,

$$\langle x,y\rangle_t := \sum_{r=1}^{t} x_r y_r = x_1y_1 + x_2y_2 + ... + x_ty_t.$$

One can easily check that the "inner product" $\langle\cdot,\cdot\rangle_t$ satisfies the Cauchy-Schwartz inequality, i.e.,

$$|\langle x,y\rangle_t| \leq |x|_t \,.\, |y|_t, \quad x,y \in \mathbb{K}^t.$$

In contrast with the classical setting, the norm $|\cdot|_t$ does not stem from the "inner product" $\langle\cdot,\cdot\rangle_t$.

Each element $x = (x_1, x_2, x_3, ...., x_t)$ in $\mathbb{K}^t$ can be written as $x = \sum_{s=1}^{t} x_r e_r$, where $e_r = (\underbrace{0,0,0,...,1}, ...,0,0,0)$ with 1 at the $r$th term and zeros elsewhere. It is clear that such a decomposition is unique. The (finite) sequence $(e_r)_{r=1,...,t}$ will be called orthonormal base of $\mathbb{K}^t$.

## 1.3 Construction of $\mathbb{Q}_p$

### 1.3.1 Introduction

The $p$-adic numbers were discovered by Hensel at the end of the 19th century as a tool in number theory. Today, those numbers play a key role in many areas beyond number theory – among those areas are algebraic geometry, analysis, $p$-adic physics, $p$-adic quantum mechanics, representation theory, and many others. In what follows, we describe, by the means of completion arguments, the construction of $\mathbb{Q}_p$, the field of $p$-adic numbers. Nevertheless, one should point out that we will omit proofs of the classical results related to such a construction and refer the reader to the landmark book by Gouvêa**[33]**, which contains a comprehensive presentation on $p$-adic numbers and related issues.

### 1.3.2 The Field $\mathbb{Q}_p$

**Definition 3.** Let $p \in \mathbb{Z}$ be a prime. A $p$-adic valuation on $\mathbb{Z}$ is a function $v_p : \mathbb{Z} - \{0\} \mapsto \mathbb{R}$, where $v_p(t)$ is the unique positive integer such that $t = p^{v_p(t)}.q$, where $q$ is not divisible by $p$.

Note that if $v_p$ is a $p$-adic valuation on $\mathbb{Z}$, then it can be extended to the field of rational numbers $\mathbb{Q}$ as follows: If $x = \frac{t}{s} \in \mathbb{Q} - \{0\}$, where $t, s \in \mathbb{Z} - \{0\}$, one sets, $v_p(x) = v_p(t) - v_p(s)$, and if $x = 0$, then one sets $v_p(0) = \infty$.

**Definition 4.** Let $x \in \mathbb{Q}$. The $p$-adic absolute value of $x$ is defined by

$$|x| := \begin{cases} p^{-v(x)} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

where $v(x)$ is the valuation of $x \in \mathbb{Q}$.

It is not hard to check that $(\mathbb{Q}, |\cdot|)$ is a non-Archimedean valued field. Moreover, from $|\cdot|$, the $p$-adic absolute value, one defines the $p$-adic metric $d$ on $\mathbb{Q} \times \mathbb{Q}$ by setting:

$$d(x,y) = |x - y|$$

for all $x, y \in \mathbb{Q}$.

**Proposition 6.** *The non-Archimedean valued field $(\mathbb{Q}, |\cdot|)$ is not complete, i.e., there exists at least one Cauchy sequence in $(\mathbb{Q}, |\cdot|)$, which does not converge for $|\cdot|$, the p-adic absolute value.*

*Proof.* See details of the proof in Gouvêa **[33]**.

**Definition 5.** The completion of $(\mathbb{Q}, |\cdot|)$ is called the field of $p$-adic numbers and denoted by $(\mathbb{Q}_p, |\cdot|)$.

The following characterization of $p$-adic numbers is due to Hensel[4]: each $x \in \mathbb{Q}_p$ can (uniquely) be expressed as

$$x = \sum_{t \geq -t_0} a_t p^t,$$

where $0 \leq a_t \leq p-1$.

Moreover, $v_p(x) = -t_0$, that is, $|x| = p^{t_0}$.

*Example 1.* In $(\mathbb{Q}_p, |\cdot|)$ ($p \geq 2$ being a prime) consider the sequence $x_t = 1 + p^2 + \ldots + p^t$. Clearly, $|x_t| = 1$ for each $t \in \mathbb{N}$. Moreover,

$$\lim_{t \to \infty} \left| (1 + p^2 + \ldots + p^t) - \frac{1}{1-p} \right| = 0.$$

Set $p = 2$. In contrast with the classical case where $\sum_{t=0}^{\infty} 2^t = \infty$, in $\mathbb{Q}_2$, such a series converges to $-1$, that is,

$$1 + 4 + 8 + 16 + \ldots + \ldots. = -1 \text{ in } \mathbb{Q}_2.$$

*Example 2.* In $(\mathbb{Q}_p, |\cdot|)$ ($p \geq 2$ being a prime) consider the sequence $x_t = 1 + p^{-2} + \ldots + p^{-t}$. Clearly, $|x_t| = p^t$ for each $t \in \mathbb{N}$. Moreover,

$$\lim_{t \to \infty} \left| 1 + p^{-2} + \ldots + p^{-t} \right| = \infty.$$

**Definition 6.** The valuation ring $\Omega_1 = \Omega(0,1)$ of $(\mathbb{Q}_p, |\cdot|)$ is denoted by $\mathbb{Z}_p$ and given by

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : x = \sum_{t \geq 0} a_t p^t, \ \ 0 \leq a_t \leq p-1\}.$$

Note that the unit ball $\mathbb{Z}_p$ is also called ring of $p$-adic integers.

We have the following characterization of $\mathbb{Z}_p$:

**Proposition 7.** *The set of p-adic integers $\mathbb{Z}_p$ is a subring of $\mathbb{Q}_p$. Moreover, each element of $\mathbb{Z}_p$ is the limit of a sequence of nonnegative integers and conversely, each Cauchy sequence in $\mathbb{Q}$ consisting of integers has a limit in $\mathbb{Z}_p$.*

*Proof.* See Baker **[3]**.

[4] This characterization is also known as the Hensel decomposition.

### 1.3.3 Convergence of Power Series over $\mathbb{Q}_p$

This subsection examines the convergence of power series in the field $(\mathbb{Q}_p, |\cdot|)$ of $p$-adic numbers.

Consider the power series

$$f(x) = \sum_{t\in\mathbb{N}} \alpha_t x^t, \quad x \in \mathbb{Q}_p, \tag{1.3.1}$$

where $(\alpha_t)_{t\in\mathbb{N}} \subset \mathbb{Q}_p$.

First of all, consider the particular case when $x = 1$ in (1.3.1), that is, the series

$$\sum_{t\in\mathbb{N}} \alpha_t, \quad \alpha_t \in \mathbb{Q}_p. \tag{1.3.2}$$

**Proposition 8.** *The series in (1.3.2) converges in $\mathbb{Q}_p$ if and only if $|\alpha_t| \to 0$ as $t \to \infty$.*

*Proof.* This is a straightforward consequence of Proposition 3.

More generally, we want find some (necessary) and sufficient conditions so that the series given in (1.3.1) converges, i.e., $\lim_{t\to\infty} |\alpha_t| . |x^t| = 0$.

**Definition 7.** A number $r$ is called the radius of convergence of the series in (1.3.1) if it converges whenever $|x| \le r$ and diverges, otherwise, i.e., $|x| > r$.

As in the classical context, define the radius of convergence of $f$ as

$$r := \frac{1}{\overline{\lim} \sqrt[t]{|\alpha_t|_p}}.$$

**Proposition 9.** *Let $f(x)$ be the series given in (1.3.1). If $r = 0$, then $f(x)$ converges at $x = 0$ only. If $r = \infty$, the series converges for each $x \in \mathbb{Q}_p$. Now if $0 < r < \infty$ and $\lim_{t\to\infty} |a_t| r^t = 0$, then $f(x)$ converges if and only if $|x| \le r$. Lastly, if $0 < r < \infty$ and $\lim_{t\to\infty} |a_t| r^t \neq 0$, then the series $f(x)$ converges if and only if $|x| < r$.*

*Proof.* See details in **[33]**.

*Example 3.* Consider the power series

$$f(x) = \sum_{t\in\mathbb{N}} x^t \ \text{ for each } \ x \in \mathbb{Q}_p.$$

One can easily check that $r = 1$ and $\lim_{t\to\infty} |a_t| r^t \neq 0$, and hence $f$ converges for each $x \in \mathbb{Q}_p$ such that $|x| < 1$, i.e., $x \in \Omega'_1$.

*Example 4.* Consider the power series ($a_t = 1$):

$$f(x) = \sum_{t\in\mathbb{N}} \frac{x^t}{p^t} \ \text{ for each } \ x \in \mathbb{Q}_p.$$

One can easily see that $r = 1/p$ and $\lim_{t\to\infty} |a_t| r^t = 1 \neq 0$, and hence the power series converges if and only if $|x| < 1/p$, i.e., $x \in \Omega'_{1/p}$.

Other important power series are that of the $p$-adic logarithm and exponential. In contrast with the classical setting, the $p$-adic exponential is not defined and analytic everywhere in $\mathbb{Q}_p$.

**Definition 8.** Let $\Omega'(1,1) = \{x \in \mathbb{Z}_p : |x-1| < 1\} = 1 + p\mathbb{Z}_p$. The $p$-adic logarithm is the power series defined, from $\Omega(1,1)$ into $\mathbb{Q}_p$, by

$$\log_p(x) := \sum_{t\in\mathbb{N}} (-1)^{t+1} \frac{(x-1)^t}{t} \quad \text{for each } \ x \in \Omega'(1,1). \tag{1.3.3}$$

**Definition 9.** Let $\Omega'_{p^{\frac{-p}{p-1}}} = \{x \in \mathbb{Z}_p : |x| < p^{\frac{-p}{p-1}}\}$. The $p$-adic exponential is the power series defined, from $\Omega'_{p^{\frac{-p}{p-1}}}$ into $\mathbb{Q}_p$, by

$$\exp_p(x) := \sum_{t\in\mathbb{N}} \frac{x^t}{t!} \quad \text{for each } \ x \in \Omega'_{p^{\frac{-p}{p-1}}}. \tag{1.3.4}$$

It can be checked that

$$\exp_p(x+y) = \exp_p(x)\exp_p(y)$$

whenever $x+y \in \Omega'_{p^{\frac{-p}{p-1}}}$.

Moreover, $\exp_p(x)$ belongs to $1+p\mathbb{Z}_p$, the domain of $\log_p$, with

$$\log_p(\exp_p x) = x$$

for each $x \in \Omega'_{p^{\frac{-p}{p-1}}}$.

Similarly, $\log_p(x+1)$ belongs to $\Omega'_{p^{\frac{-p}{p-1}}}$ with

$$\exp_p(\log_p(x+1)) = x+1.$$

## 1.4 Construction of $\mathbb{K}((x))$

Let $\mathbb{K}$ be any field (possibly $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}_p$) and let $\mathbb{K}[x]$ be the ring of polynomials whose coefficients belong to $\mathbb{K}$. Basically, each $P \in \mathbb{K}[x]$ can be expressed as:

$$P(x) = \sum_{t=0}^{n} a_t x^t,$$

where $a_t \in \mathbb{K}$ for $t = 1,2,\ldots,n$. In that event, one can write $P(x) = x^{\nu(P)}.Q(x)$, where $Q \in \mathbb{K}[x]$ with $Q(0) \neq 0$. It is not hard to check that

(1) $\nu(P.Q) = \nu(P) + \nu(Q)$, for all $P, Q \in \mathbb{K}[x]$;

(2) $v(P+Q) \geq \min(v(P), v(Q))$, for all $P, Q \in \mathbb{K}[x]$.

Setting $v(0) = +\infty$ and using (1)-(2) above, one can easily see that $v$ is a valuation [5] over the ring $\mathbb{K}[x]$. One can extend such a valuation over the field $\mathbb{K}(x)$ of rational fractions by setting: For all $f = \dfrac{P}{Q} \in \mathbb{K}(x)$,

$$v(f) = v(P) - v(Q).$$

Now for $0 < \varsigma < 1$, one sets, $|P| = \varsigma^{v(P)}$ for each $P \in \mathbb{K}[x]$. It is not hard to check that $|\cdot|$ is a non-Archimedean absolute value, which can be extended over $\mathbb{K}(x)$. The non-Archimedean field $(\mathbb{K}(x), |\cdot|)$ is not complete; let $(\mathbb{K}((x)), |\cdot|)$ denote its completion. The field $(\mathbb{K}((x)), |\cdot|)$ is then called the field of formal Laurent series.

*Example 5.* In $\mathbb{Q}_p[x]$ ($p \geq 2$ is a prime), let $v(L) = -deg(L)$ if $L$ is a polynomial and let $\varsigma = |p| = p^{-1}$. The valuation $v$ can be extended over $\mathbb{Q}_p((x))$ as above.

Consider $f \in \mathbb{Q}_p((x))$ defined by

$$f(x) = \frac{L(x)}{M(x)},$$

where $L(x) = a_0 + a_1 x + ... + a_n x^n$ and $M(x) = b_0 + b_1 x + ... + b_m x^m$ with $a_i, b_j \in \mathbb{Q}_p$ for $i = 0, 1, .., n,\ j = 0, 1, ..., m$, and $a_n, b_m \neq 0$. It is easy to check that

$$|f| = p^{-v(f)} = \frac{1}{p^{m-n}}.$$

## 1.5 Bibliographical Notes

This chapter introduces classical results on non-Archimedean valued fields. The uncovered topics related to non-Archimedean valued fields and related issues can be found in most of books devoted to non-Archimedean functional analysis, see, e.g., Gouvêa**[33]**, Koblitz**[42]**, Mahler**[50]**, Ochsenius & Schikhof**[56]**, Robert**[60]**, van Rooij**[62]**, Vladimirov, Volovich & Zelenov**[66]**, and many others.

Basic tools of Subsection 1.3.3, that is, the convergence of power series were taken in both Gouvêa**[33]** and Vladimirov, Volovich & Zelenov**[66]**. Similarly, basic and advanced tools on the algebra of analytic functions in the non-Archimedean setting can be found in those books. Section **1.4** related to the field of formal Laurent series was taken in Diarra**[25]**.

[5] Among possible valuations that can be defined over $\mathbb{K}[x]$, one may consider that defined through the degree of the polynomial, i.e., if $P \in \mathbb{K}[x]$, one defines $v_\infty(P) = -deg(P)$.

*Chapter 2*

# Non-Archimedean Banach and Hilbert Spaces

## 2.1 Non-Archimedean Banach Spaces

This section introduces non-Archimedean (free) Banach and Hilbert spaces. As for non-Archimedean valued fields, those Banach and Hilbert spaces play a crucial role in the composition of this book. For uncovered topics related to those non-Archimedean Banach spaces, we refer the reader to the excellent book by Rooij **[62]**, which covers all topics related to those spaces. Moreover, for all questions on non-Archimedean Hilbert spaces, which are not treated here, we refer to the work of Diarra**[24, 25]**.

Throughout the rest of this section, $(\mathbb{K}, \|\cdot\|)$ denotes a complete non-Archimedean field.

### 2.1.1 Basic Definitions

**Definition 10.** Let $\mathbb{X}$ be a vector space over $\mathbb{K}$. A nonnegative real valued function $\|\cdot\| : \mathbb{X} \mapsto [0,\infty)$ is called a norm if:

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{K}$ and $x \in \mathbb{X}$; and
(3) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{X}$.

The norm $\|\cdot\|$ is called non-Archimedean if the statement (3) above can be replaced by the stronger condition

(4) $\|x+y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in \mathbb{X}$.

As for the absolute value of a non-Archimedean valued field, (4) above is called non-Archimedean triangle inequality. A normed vector space $(\mathbb{X}, \|\cdot\|)$ satisfying [Definition 10, (1)-(2)-(4)] is called a non-Archimedean normed vector space[1]. Moreover, as an immediate consequence of [Definition 10, (4)], the following holds:

$$\|x+y\| = \max(\|x\|, \|y\|) \quad \text{whenever} \quad \|x\| \neq \|y\|.$$

More generally,

[1] A non-Archimedean normed vector space $(\mathbb{X}, \|\cdot\|)$ is also called ultrametric normed vector space.

**Proposition 10.** *Let $(\mathbb{X}, \|\cdot\|)$ be a non-Archimedean normed vector space over $\mathbb{K}$. If $x_1, \ldots x_t \in \mathbb{X}$ such that $\|x_s\| \neq \|x_r\|$ for all $s \neq r$, then*

$$\|x_1 + x_2 + \ldots + x_t\| = \max_{1 \le s \le t} \|x_s\|.$$

**Proposition 11.** *Let $(\mathbb{X}, \|\cdot\|)$ be a non-Archimedean normed vector space over $\mathbb{K}$. Then*[2]

$$\|x+y\|^2 + \|x-y\|^2 \le 2\max(\|x\|^2, \|y\|^2)\ x, y \in \mathbb{X}, \tag{2.1.1}$$

*for all $x, y \in \mathbb{X}$, with equality whenever $\|x\| \neq \|y\|$.*

**Definition 11.** A non-Archimedean Banach space is a non-Archimedean normed vector space, which is complete.

### 2.1.2 Examples of Non-Archimedean Banach Spaces

*Example 6.* Let $(\omega_t)_{t \in I} \subset \mathbb{R}^+ - \{0\}$ be a family of nonzero real numbers [3]. Define the $\mathbb{K}$-vector space $B^\infty(I, \mathbb{K}, \omega)$ by

$$B^\infty(I, \mathbb{K}, \omega) := \{x = (x_t)_{t \in I} \in \mathbb{K}^I : \sup_{t \in I} |x_t| \omega_t < \infty\}.$$

The space $B^\infty(I, \mathbb{K}, \omega)$ is equipped with the sup norm defined by

$$\|x\| := \sup_{t \in I} |x_t| \omega_t.$$

It is not hard to see that $(B^\infty(I, \mathbb{K}, \omega), \|\cdot\|)$ is a non-Archimedean Banach space.

*Example 7.* Let $c_0(I, \mathbb{K}, \omega) \subset (B^\infty(I, \mathbb{K}, \omega)$ be the subspace defined by

$$c_0(I, \mathbb{K}, \omega) := \{x = (x_t)_{t \in I} \in \mathbb{K}^I : \lim_{t \in I} |x_t| \omega_t = 0\}.$$

Clearly $(c_0(I, \mathbb{K}, \omega), \|\cdot\|)$ is a closed subspace of $(B^\infty(I, \mathbb{K}, \omega), \|\cdot\|)$, and hence is also a non-Archimedean Banach space.

*Example 8.* Let $M$ be a compact (topological) space. Define the space of all continuous functions which go from $M$ into $\mathbb{K}$ by

$$C(M, \mathbb{K}) := \{u : M \mapsto \mathbb{K},\ \ u \text{ is continuous}\}.$$

We then equip $C(M, \mathbb{K})$ with the sup norm: $\|u\|_\infty := \sup_{t \in M} |u(t)|$.

**Theorem 2.** *The normed vector space $(C(M, \mathbb{K}), \|\cdot\|_\infty)$ defined above is a non-Archimedean Banach space.*

[2] Such an inequality is known as the non-Archimedean Parallelogram Law in a normed vector space.

[3] $I$ is the index set. It is customary to take $I = \mathbb{N}$. However the treatment of those non-Archimedean Banach space in the general case can be found in Diarra**[24, 25]**.

## 2.2 Free Banach Spaces

### 2.2.1 Definitions

**Definition 12.** A non-Archimedean Banach space $(\mathbb{X}, \|\cdot\|)$ over $\mathbb{K}$ is said to be a *free* Banach space if there exists a family $(e_t)_{t\in I}$ of elements of $\mathbb{X}$ such that each element $x \in \mathbb{X}$ can be written in a unique fashion as a pointwise convergent series defined by

$$x = \sum_{t\in I} x_t e_t \quad \text{with} \quad \lim_{t\in I} x_t e_t = 0,$$

and $\|x\| = \sup_{t\in I} |x_t| \, \|e_t\|$. The family $(e_t)_{t\in I}$ is then called an *orthogonal base* for $\mathbb{X}$. If $\|e_t\| = 1$, for all $t \in I$, then $(e_t)_{t\in I}$ is called an *orthonormal base* for $\mathbb{X}$.

### 2.2.2 Examples

*Example 9.* In Example **8**, take $M = \mathbb{Z}_p$ and $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$ where $p \geq 2$ is a prime number. Then the resulting space $(C(\mathbb{Z}_p, \mathbb{Q}_p), \|\cdot\|_\infty)$ is a free Banach space. Indeed, consider the sequence of functions defined by:

$$f_t(x) = \frac{x(x-1)(x-2)(x-3)....(x-t+1)}{t!}, \quad t \geq 1,$$
$$f_0(x) = 1.$$

It is well-known **[46]** that the family $(f_t)_{t\in\mathbb{N}}$ is an orthonormal base, i.e., $\|f_t\|_\infty = 1$. Moreover, for each $u \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ has a unique uniformly convergent decomposition defined by

$$u(x) = \sum_{t=0}^{\infty} c_t f_t(x), \quad c_t \in \mathbb{Q}_p$$

with $|c_t| \mapsto 0$ as $t \mapsto \infty$ and $\|u\|_\infty = \sup_{t\in\mathbb{N}} |c_t|$.

Let $(\mathbb{X}, \|\cdot\|)$ be a free Banach space over $\mathbb{K}$. If $\mathbb{X}^*$ denotes the (topological) dual of $\mathbb{E}$, then the following spaces are isomorphic:

$$\mathbb{X} \simeq c_0(I, \mathbb{K}, (\|e_t\|_{t\in I})), \quad \text{and} \quad \mathbb{X}^* \simeq B^\infty(I, \mathbb{K}, (\frac{1}{\|e_t\|})_{t\in I}).$$

*Example 10.* The space $c_0(I, \mathbb{K}, \omega)$ given in Example 7 is a free Banach space (see details in the next subsection).

*Example 11.* Let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean field. Then the $t$-vector space $\mathbb{K}^t$ previously defined is a free Banach.

*Remark 1.* Let $(e_t)_{t\in I}$ be an orthogonal base for the free Banach space $(\mathbb{X}, \|\cdot\|)$. Define $e'_t \in \mathbb{X}^*$ (topological dual of $\mathbb{X}$) by:

$$x = \sum_{t\in I} x_t e_t, \;\; e'_t(x) = x_t.$$

It turns out that $\|e'_t\| = \dfrac{1}{\|e_t\|}$. Furthermore, each $x' \in \mathbb{X}^*$ can be expressed as a pointwise convergent series defined by

$$x' = \sum_{t\in I} \langle x', e_t\rangle e'_t, \;\; \text{and}$$

$$\|x'\| = \sup_{t\in I} \frac{|\langle x', e_t\rangle|}{\|e_t\|}.$$

## 2.3 Non-Archimedean Hilbert Spaces

### 2.3.1 Introduction

This section is devoted to the so-called non-Archimedean Hilbert spaces $\mathbb{E}_\omega$. They play a key role throughout the book. Among other things, we will see that the norm $\|\cdot\|$ of a non-Archimedean Hilbert space $\mathbb{E}_\omega$ does not stem from its corresponding inner product $\langle \cdot, \cdot \rangle$. Moreover, it may happen that $|\langle x, x\rangle| < \|x\|^2$ for some $x \in \mathbb{E}_\omega$. In addition to that $\mathbb{E}_\omega$ contains isotropic vectors $x \neq 0$, that is, $\langle x, x\rangle = 0$. For more on these spaces and related issues we refer the reader to **[4]**, **[24]**, and **[25]**.

### 2.3.2 Non-Archimedean Hilbert Spaces

Let $\omega = (\omega_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of non-zero elements and define the space $\mathbb{E}_\omega = c_0(\mathbb{N}, \mathbb{K}, \omega)$, that is,

$$\mathbb{E}_\omega := \{x = (x_t)_{t\in\mathbb{N}},\ \forall t,\ x_t \in \mathbb{K} \text{ and } \lim_{t\to\infty} |x_t||\omega_t|^{1/2} = 0\}.$$

It can be easily shown that $x = (x_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega$ if and only if

$$\lim_{t\to\infty} x_t^2 \omega_t = 0.$$

Moreover, $\mathbb{E}_\omega$ is a non-Archimedean Banach space over $\mathbb{K}$ when it is endowed with the norm

$$x = (x_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega, \quad \|x\| = \sup_{t\in\mathbb{N}} |x_t||\omega_t|^{1/2}. \tag{2.3.1}$$

It is also clear that $\mathbb{E}_\omega$ is a free Banach space and has a *canonical orthogonal base*, namely, $(e_t)_{t\in\mathbb{N}}$, where $e_t$ is the sequence whose terms are 0 except the $t$-th term, which is 1, in other words, $e_t = (\delta_{ts})_{s\in\mathbb{N}}$, where $\delta_{ts}$ is the Kronecker symbol. We shall make extensive use of this canonical orthogonal base throughout the rest of this book. For each $t$, $\|e_t\| = |\omega_t|^{1/2}$. If $|\omega_t| = 1$, we shall refer to $(e_t)_{t\in\mathbb{N}}$ as the canonical orthonormal base of $\mathbb{E}_\omega$.

Let $\langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \to \mathbb{K}$ be the linear form defined for all $u,v \in \mathbb{E}_\omega$, $u = (u_t)_{t\in\mathbb{N}}$, $v = (v_t)_{t\in\mathbb{N}}$, by

$$\langle u,v\rangle := \sum_{t\in\mathbb{N}} \omega_t u_t v_t. \tag{2.3.2}$$

Clearly, $\langle\cdot,\cdot\rangle$ is a symmetric, bilinear, and non-degenerate linear form on $\mathbb{E}_\omega$ and satisfies the Cauchy-Schwarz's inequality:

$$\forall u,v \in \mathbb{E}_\omega,\ |\langle u,v\rangle| \ \leq\ \|u\|\ .\ \|v\|. \tag{2.3.3}$$

*Remark 2.* The $\mathbb{K}$-form $\langle\cdot,\cdot\rangle$ defined in (2.3.2) is called non-Archimedean inner product.

On the canonical orthogonal base, the following holds:

$$\langle e_t, e_s\rangle = \omega_t \delta_{ts} = \begin{cases} 0 & \text{if } t \neq s \\ \omega_t & \text{if } t = s. \end{cases}$$

**Definition 13.** The space $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$, where $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ are respectively the norm defined in (2.3.1) and the inner product in (2.3.2), is called a non-Archimedean Hilbert space.

**Proposition 12.** *Let $\omega = (\omega_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of nonzero terms and let $\mathbb{E}_\omega$ be the corresponding non-Archimedean space. If $(e_t)_{t\in\mathbb{N}}$ denotes the canonical orthogonal base of $\mathbb{E}_\omega$, then, for all $x \in \mathbb{E}_\omega$,*

$$\left|\sum_{t=0}^{\infty} \frac{(<x,e_t>)^2}{\omega_t}\right| \leq \|x\|^2. \tag{2.3.4}$$

*Proof.* Observe that $\dfrac{(<x,e_t>)^2}{\omega_t} = \dfrac{\omega_t^2 x_t^2}{\omega_t} = \omega_t x_t^2$. Moreover, $\lim_{t\to\infty}|\omega_t x_t^2| = 0$, by $x = \sum_{t\in N} x_t e_t \in \mathbb{E}_\omega$ ($\lim_{t\to\infty} |\omega_t|^{1/2}.|x_t| = 0$). Hence, the series $\sum_{t=0}^{\infty} \dfrac{(<x,e_t>)^2}{\omega_t}$ converges.

Now

$$\begin{aligned}
\left|\sum_{t=0}^{\infty} \frac{(<x,e_t>)^2}{\omega_t}\right| &= \left|\sum_{t=0}^{\infty} \omega_t x_t^2\right| \\
&\leq \sup_{t\geq 0} |\omega_t||x_t|^2 \\
&= \sup_{t\geq 0} (|\omega_t|^{1/2}|x_t|)^2 \\
&= (\ \sup_{t\geq 0} |\omega_t|^{1/2}|x_t|\ )^2 \\
&= \|x\|^2.
\end{aligned}$$

Consequently,

**Corollary 1.** *If $\omega = (\omega_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ where $\omega_t = 1_{\mathbb{K}}$ for each $t \in \mathbb{N}$ and if $\mathbb{E}_\omega$ is the corresponding non-Archimedean space, then, for all $x \in \mathbb{E}_\omega$,*

$$\left|\sum_{t=0}^{\infty}(< x, e_t >)^2\right| \leq \|x\|^2.$$

Another example of non-Archimedean Hilbert space consists of considering $C(\mathbb{Z}_p, \mathbb{Q}_p)$ as in Example 9 and put the inner product (symmetric, bilinear, non degenerate form) is defined as follows: for all $u = \sum_{t=0}^{\infty} u_t f_t$, and $v = \sum_{t=0}^{\infty} v_t f_t$ in $C(\mathbb{Z}_, \mathbb{Q}_p)$,

$$\langle u, v\rangle := \sum_{t=0}^{\infty} u_t v_t.$$

It is clear that the inner product $\langle\cdot,\cdot\rangle$ given above is well-defined as both $u_t$ and $v_t \mapsto 0$ in $\mathbb{Q}_p$ as $t \to \infty$, and the Cauchy-Schwartz inequality holds

$$|\langle u, v\rangle| \leq \|u\|_\infty . \|v\|_\infty.$$

In view of the above, it easily follows that $\langle f_t, f_s\rangle = \delta_{ts}$ where $\delta_{ts}$ are the classical Kronecker symbols.

The triplet $(C(\mathbb{Z}_p, \mathbb{Q}_p), \|\cdot\|_\infty, \langle\cdot,\cdot\rangle)$ will be called a $p$-adic or non-archimedean Hilbert space. One must point out that the norm on $C(\mathbb{Z}_p, \mathbb{Q}_p)$ does not stem from its corresponding inner product.

### 2.3.3 The Hilbert Space $\mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2} \times \ldots \times \mathbb{E}_{\omega_t}$

Let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean field with characteristic $char(\mathbb{K})$ zero. Examples of such fields include $\mathbb{Q}_p$ ($p \geq 2$ being prime) the field of $p$-adic numbers.

Let $\omega_\tau = (\omega_s^\tau)_{s\in\mathbb{N}}$ with $\tau = 1, 2, \ldots, t$ be ($t$ being fixed) $t$ sequences of nonzero terms in $\mathbb{K}$. For each $\tau$ ($\tau = 1, 2, \ldots t$) we consider the corresponding non-Archimedean Hilbert space $\mathbb{E}_{\omega_\tau}$ equipped with its usual topologies (see (2.3.1)-(2.3.2)).

Let $\mathbb{E}_t$ denote the direct sum $\mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2} \times \ldots \times \mathbb{E}_{\omega_t}$ of $\mathbb{E}_{\omega_1}$, $\mathbb{E}_{\omega_2}$, ..., and $\mathbb{E}_{\omega_t}$, respectively. We now equip the vector space $\mathbb{E}_t$ with the non-Archimedean norm defined by

$$||(u_1, \ldots, u_t)||_n := \max(\|u_1\|, \ldots, \|u_t\|), \tag{2.3.5}$$

for all $(u_1, \ldots, u_t) \in \mathbb{E}_t$.

Arguing that each $(\mathbb{E}_{\omega_\tau}, \|.\|)$ for $\tau = 1, 2, \ldots t$ is a non-Archimedean Banach space it follows that $(\mathbb{E}_t, ||\cdot||_t)$ is also a non-Archimedean Banach space. It also clear that $\mathbb{E}_t$ ($t$ fixed) is a free Banach space with orthogonal base: $\{(\mathcal{F}_s^\tau)_{s\in\mathbb{N}},\ \tau = 1, 2, \ldots, t\}$ where $\mathcal{F}_s^\tau = (\underbrace{0, 0, \ldots e_s^\tau}, ..0, 0, 0)$ whose terms are 0 except the $s$-th term, which is $e_s^\tau$ ($(e_s^\tau)_{s\in\mathbb{N}}$ being the canonical orthogonal base of $\mathbb{E}_{\omega_\tau}$ for $\tau = 1,2,..,t$). One can easily check that $\|\mathcal{F}_s^\tau\|_t = \|e_s^\tau\| = |\omega_s^\tau|^{1/2}$ for all $s \in \mathbb{N}$ and for each $\tau = 1,2,...,t$. We confer to $\{(\mathcal{F}_s^\tau)_{s\in\mathbb{N}},\ \ \tau = 1,2,...,t\}$ as the canonical orthogonal base for $\mathbb{E}_t$.

Let $\langle\cdot,\cdot\rangle_t : \mathbb{E}_t \times \mathbb{E}_t \mapsto \mathbb{K}$ be the $\mathbb{K}$-bilinear form defined by

$$\langle(u_1,...,u_t),(v_1,...,v_t)\rangle_t := \sum_{\tau=1}^{t} \langle u_\tau, v_\tau\rangle, \tag{2.3.6}$$

for all $(u_1,...,u_t),(v_1,...,v_t) \in \mathbb{E}_t$ where $\langle\cdot,\cdot\rangle$ is the inner product of each $\mathbb{E}_{\omega_\tau}$ for $\tau = 1,2,...,t$. Then, $\langle\cdot,\cdot\rangle_t$ is a symmetric, non-degenerate form on $\mathbb{E}_t \times \mathbb{E}_t$.

**Definition 14.** The space $\mathbb{E}_t := \mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2} \times ... \times \mathbb{E}_{\omega_t}$ endowed with the norm given in (2.3.5) and the $\mathbb{K}$-bilinear form $\langle\cdot,\cdot\rangle_t$ given in (2.3.6) is called a non-Archimedean Hilbert space.

**Proposition 13.** *For every $(u_1,...,u_t),(v_1,...,v_t) \in \mathbb{E}_t$, the Cauchy-Schwarz inequality holds, that is,*

$$|\langle(u_1,...,u_t),(v_1,...,v_t)\rangle_t| \leq ||(u_1,...,u_t)||_n \,.\, ||(v_1,...,v_t)||_t.$$

*Proof.* Suppose $t = 2$. For all $(u,v),\ (x,y) \in \mathbb{E}_{\omega_1} \times \mathbb{E}_{\omega_2}$,

$$\begin{aligned}
|\langle(u,v),(x,y)\rangle_2| &= |\langle u,x\rangle + \langle v,y\rangle| \\
&\leq \max(|\langle u,x\rangle|, |\langle v,y\rangle|) \\
&\leq \max(\|u\|\ \|x\|, \|v\|\ \|y\|) \\
&\leq \max(||(u,v)||_2\ \|x\|, ||(u,v)||_2\ \|y\|) \\
&\leq \max(||(u,v)||_2\ ||(x,y)||_2, ||(u,v)||_2\ ||(x,y)||_2) \\
&= ||(u,v)||_2\ ||(x,y)||_2.
\end{aligned}$$

When $t \geq 3$, one follows the same lines as above.

If $M \subset \mathbb{E}_\omega \times \mathbb{E}_\omega$ is a subspace, then its orthogonal complement $M^\perp$ with respect the $\mathbb{K}$-bilinear form $\langle\cdot,\cdot\rangle_2$ is defined by

$$M^\perp := \{(u,v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega :\ \langle(u,v),(x,y)\rangle_2 = 0,\ \forall (x,y) \in M\}.$$

One can easily check that $M^\perp$ is closed. This is actually a consequence of the continuity of the bilinear form defined by $\Phi_{(x,y)} : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ and

$$\Phi_{(x,y)}(u,v) = \langle(u,v),(x,y)\rangle_2,\ \ \forall (u,v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega,$$

where $(x,y) \in \mathbb{E}_\omega \times \mathbb{E}_\omega$ is fixed.

## 2.4 Bibliographical Notes

This chapter is entirely devoted to non-Archimedean Banach spaces, free Banach spaces, and non-Archimedean Hilbert spaces. Algebraic properties of those spaces can be found in Ochsenius & Schikhof **[56]**, and van Rooij **[62]**. For more on free Banach spaces and non-Archimedean Hilbert spaces, we refer the reader to Diagana **[17]**, especially Diarra **[24, 25]**, and Khrennikov **[37]**. Details on the Banach space $C(\mathbb{Z}_p, \mathbb{Q}_p)$ can be found in Kochubei **[46]**. Subsection **2.3.3** was taken in Diagana **[17]**.

*Chapter 3*

# Non-Archimedean Bounded Linear Operators

## 3.1 Introduction

This chapter studies the class of bounded linear operators on non-Archimedean (free) Banach and Hilbert spaces. Section 3.2 examines bounded linear operators on an arbitrary non-Archimedean Banach space while Section 3.3 considers those operators on $\mathbb{E}_\omega$, in particular, the existence of an adjoint operator for a given bounded linear operator is screened. Section 3.4 presents some recent results on perturbation of bases of $\mathbb{E}_\omega$ obtained by Diagana & Ramaroson in **[12]**. Section 3.5 is concerned with non-Archimedean analogues of the classical *Hilbert-Schmidt* operators as well as their properties (Diagana et al. **[4]**). Some of the results go along the classical line and others deviate from it. For the most part, the statements of the results are inspired by their classical counterparts, however, their proofs may depend heavily on the non-Archimedean nature of $\mathbb{E}_\omega$ and the ground field $\mathbb{K}$. Among other things, the definition of a Hilbert-Schmidt operator in the non-archimedean context depends on the base; further, Hilbert-Schmidt operator has an adjoint, which is again a Hilbert-Schmidt operator and that both have the same Hilbert-Schmidt norm. $B_2(\mathbb{E}_\omega)$, the collection of all Hilbert-Schmidt operators on $\mathbb{E}_\omega$ is a two-sided ideal in the ring $B_0(\mathbb{E}_\omega)$ of all bounded operators, which have adjoint. In addition to that it will be shown that every Hilbert-Schmidt operator is *completely continuous* in the sense that it is a limit, in $B(\mathbb{E}_\omega)$, of a sequence of operators of finite ranks. As in the classical setting, a natural inner product is considered for Hilbert-Schmidt operators and we show that it satisfies the Cauchy-Schwarz inequality. We define the trace of an operator. As in the classical context, the trace may or may not exist. It is then shown that a Hilbert-Schmidt operator has a trace. We next illustrate those abstract results with several examples at the end of Section 3.5.

## 3.2 Bounded Linear Operators on Non-Archimedean Banach Spaces

Throughout this section, unless otherwise, $(\mathbb{K}, |\cdot|)$ denotes a complete non-Archimedean field.

### 3.2.1 Basic Definitions

Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be non-Archimedean Banach spaces over $\mathbb{K}$, respectively. A linear operator $A : (\mathbb{X}, \|\cdot\|_{\mathbb{X}}) \mapsto (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ is a transformation, which maps linearly $\mathbb{X}$ into $\mathbb{Y}$.

**Definition 15.** A linear operator $A : \mathbb{X} \mapsto \mathbb{Y}$ is said to be bounded if there exists $K \geq 0$ such that $\|Au\|_{\mathbb{Y}} \leq K\ .\ \|u\|_{\mathbb{X}}$ for each $u \in \mathbb{X}$.

The collection of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ is denoted $B(\mathbb{X}, \mathbb{Y})$. When $\mathbb{X} = \mathbb{Y}$, this is simply denoted $B(\mathbb{X})$.

*Remark 3.* If $A : \mathbb{X} \mapsto \mathbb{Y}$ is a bounded linear operator, then its norm $\|A\|$ is defined by

$$\|A\| := \sup_{u \neq 0} \left( \frac{\|Au\|_{\mathbb{Y}}}{\|u\|_{\mathbb{X}}} \right). \tag{3.2.1}$$

Clearly, $0 \in B(\mathbb{X}, \mathbb{Y})$ with $\|0\| = 0$. Furthermore, if $A, B \in B(\mathbb{X}, \mathbb{Y})$ and if $\lambda \in \mathbb{K}$, then $A + B$, $\lambda B$, $AB$ belong to $B(\mathbb{X}, \mathbb{Y})$, and the following hold:

(1) $\|A + B\| \leq \|A\| + \|B\|$;
(2) $\|\lambda B\| = |\lambda|\,.\,\|B\|$;
(3) $\|AB\| \leq \|A\|\,.\,\|B\|$.

Consequently, $(B(\mathbb{X}, \mathbb{Y}), \|\cdot\|)$ is a normed vector space.

**Lemma 1.** *The space $(B(\mathbb{X}, \mathbb{Y}), \|\cdot\|)$ of bounded linear operators is non-Archimedean normed vector space.*

*Proof.* Let $A, B \in (B(\mathbb{X}, \mathbb{Y}), \|\cdot\|)$. We want to show that the operator norm $\|\cdot\|$ satisfies the non-Archimedean inequality. Indeed,

$$\begin{aligned}
\|A + B\| &= \sup_{u \neq 0} \left( \frac{\|Au + Bu\|_{\mathbb{Y}}}{\|u\|_{\mathbb{X}}} \right) \\
&\leq \sup_{u \neq 0} \max \left( \frac{\|Au\|_{\mathbb{Y}}}{\|u\|_{\mathbb{X}}}, \frac{\|Bu\|_{\mathbb{Y}}}{\|u\|_{\mathbb{X}}} \right) \\
&= \max \left( \sup_{u \neq 0} \frac{\|Au\|_{\mathbb{Y}}}{\|u\|_{\mathbb{X}}}, \sup_{u \neq 0} \frac{\|Au\|_{\mathbb{Y}}}{\|u\|_{\mathbb{X}}} \right) \\
&= \max\left(\|A\|, \|B\|\right).
\end{aligned}$$

**Theorem 3.** *If $(\mathbb{X}, \|.\|_{\mathbb{X}})$, $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ are non-Archimedean Banach space over $\mathbb{K}$, then $(B(\mathbb{X}, \mathbb{Y}), ||\cdot||)$ is a non-Archimedean Banach space.*

The proof of Theorem 3 is similar to that of the classical setting. However, for the sake of clarity, we will provide the reader with it.

*Proof.* In view of the previous facts and Lemma 1, it remains to prove that $(B(\mathbb{X},\mathbb{Y}),\|\cdot\|)$ is complete. Let $(A_t)_{t\in\mathbb{N}} \in B(\mathbb{X},\mathbb{Y})$ be a Cauchy sequence. Therefore, for each $\forall\varepsilon > 0$ there exists $t_0(\varepsilon) \in \mathbb{N}$ such that

$$\|A_s - A_t\| \leq \varepsilon$$

whenever $s,t \geq t_0(\varepsilon)$.

Consequently, for each $x \in \mathbb{X}$, $(A_t x)_{t\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Y}$. Since $\mathbb{Y}$ is complete, $(A_t x)_{t\in\mathbb{N}}$ converges strongly in $\mathbb{Y}$ as $t \to \infty$. Define $A : \mathbb{X} \mapsto \mathbb{Y}$ by setting

$$Ax := \lim_{t\to\infty} A_t x \text{ in } \mathbb{Y}.$$

(1) *Linearity of A:* If $\lambda, \mu \in \mathbb{K}$, $\forall t \in \mathbb{N}$, then $A_t(\lambda x + \mu y) = \lambda A_t x + \mu A_t y$, for all $x,y \in \mathbb{X}$. Thus, when $t$ goes to $\infty$, it follows that $A(\lambda x + \mu y) = \lambda Ax + \mu Ay$, for all $x,y \in \mathbb{X}$.

(2) *Boundedness of A:* Fist of all, note that $\|Ax\|_{\mathbb{Y}} = \lim_{t\to\infty} \|A_t x\|_{\mathbb{Y}}$. Next, $|\|A_t\| - \|A_s\|| \leq \|A_t - A_s\| \leq \varepsilon$ whenever $s \geq t \geq t_0(\varepsilon)$, and hence $(\|A_t\|)_{t\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Clearly, $(\|A_t\|)_{t\in\mathbb{N}}$ converges, and hence is bounded, i.e., there exists $M \geq 0$ such that $\|A_t\| \leq M$ for each $t \in \mathbb{N}$. Now $\|A_t x\|_{\mathbb{Y}} \leq M \,.\, \|x\|$ for all $t \in \mathbb{N}$ and $x \in \mathbb{X}$, and hence $\|Ax\|_{\mathbb{Y}} \leq M \,.\, \|x\|$ for all $x \in \mathbb{X}$, consequently, $A$ is bounded.

To complete the proof we have to show that $\|A_t - A\| \mapsto 0$ as $t \to \infty$. In view of the above, for each $x \in \mathbb{X}$,

$$\|A_t x - A_s x\|_{\mathbb{Y}} \leq \|A_t - A_s\| \; \|x\|_{\mathbb{X}},$$

and hence

$$\|A_t x - A_s x\|_{\mathbb{Y}} \leq \varepsilon \|x\|_{\mathbb{X}} \tag{3.2.2}$$

whenever $s \geq t \geq t_0(\varepsilon)$. Letting $s \to \infty$ in (3.2.2) it follows that

$$\|A_t x - Ax\|_{\mathbb{Y}} \leq \varepsilon \|x\|_{\mathbb{X}}$$

whenever $n \geq t_0(\varepsilon)$, that is, $\|A_t - A\| \to 0$ as $t \to \infty$.

### 3.2.2 Examples

*Example 12.* Let $\mathbb{K} = \mathbb{Q}_p$ equipped with the $p$-adic absolute value and suppose that $\mathbb{X} = \mathbb{Y} = C(\mathbb{Z}_p, \mathbb{Q}_p)$, is the non-Archimedean Banach space appearing in Example 9 equipped with its corresponding sup norm. Define

$$M_\gamma : C(\mathbb{Z}_p, \mathbb{Q}_p) \mapsto C(\mathbb{Z}_p, \mathbb{Q}_p), \quad M_\gamma \phi := \gamma(x)\phi, \tag{3.2.3}$$

where $\gamma \in C(\mathbb{Z}_p, \mathbb{Q}_p)$.

In view of the above, $\|M_\gamma \phi\|_\infty \leq K \,.\, \|\phi\|_\infty$, for each $\phi \in C(\mathbb{Z}_p, \mathbb{Q}_p)$, where $K = \|\gamma\|_\infty$. Consequently, $M_\gamma$ is a bounded linear operator on $C(\mathbb{Z}_p, \mathbb{Q}_p)$. Note that $M_\gamma$ is commonly called *multiplication* operator with potential $\gamma$.

*Example 13.* Let $(\mathbb{K},|\cdot|)$ be an arbitrary (complete) non-Archimedean field and let $\mathbb{X}=\mathbb{Y}=\mathbb{K}^t$, equipped with its natural topology (see Chapter 1). Let $M=(a_{rs})_{r,s=1,..,t}$ be the $t\times t$ square matrix, i.e.,

$$Me_r=\sum_{s=1}^{t}a_{sr}e_s.$$

One can easily see that $M$ is a bounded linear operator with norm

$$\|M\|=\max_{1\leq r,s\leq t}|a_{rs}|.$$

### 3.2.3 The Banach Algebra $B(\mathbb{X})$

A vector space $(\mathbb{Y},+)$ over $\mathbb{K}$ is called an *algebra* if for each $x,y\in\mathbb{Y}$, a unique product $x*y\in\mathbb{Y}$ can be defined such that

(1) $(x*y)*z=x*(y*z)$;
(2) $x*(y+z)=x*y+x*z$;
(3) $(x+y)*z=x*z+y*z$;
(4) $\lambda(x*y)=(\lambda x)*y=x*(\lambda y)$;

for all $x,y,z\in\mathbb{Y}$ and $\lambda\in\mathbb{K}$.

An algebra $(\mathbb{Y},+,*)$ with norm $\|\cdot\|$ such that $\|x*y\|\leq\|x\|\,.\,\|y\|$ for all $x,y\in\mathbb{Y}$, is called a *normed algebra*.

A normed algebra $(\mathbb{Y},+,*,\|\cdot\|)$ is called a *Banach algebra* if it is complete for $\|\cdot\|$. Moreover, a Banach algebra $(\mathbb{Y},+,*,\|\cdot\|)$ is called a Banach algebra with unity whether it contains an element $e$ with $\|e\|=1$ such that $e*x=x*e=x$ for each $x\in\mathbb{Y}$.

In view of the above, it is clear that $(B(\mathbb{X}),+,\circ,\|\cdot\|)$, the collection of all bounded linear operators on $\mathbb{X}$ equipped with $\circ$, the composition operator, is a Banach algebra with unity.

### 3.2.4 Further Properties of Bounded Linear Operators

Let $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$, $(\mathbb{Y},\|\cdot\|_{\mathbb{Y}})$ be non-Archimedean Banach spaces over $\mathbb{K}$. If $A\in B(\mathbb{X},\mathbb{Y})$, then its kernel $N(A)$ and range $R(A)$ are respectively defined by

$$N(A)=\{x\in\mathbb{X}:Ax=0\},\ \ \text{and}$$

$$R(A)=\{Ax\in\mathbb{Y}:x\in\mathbb{X}\}.$$

Note that both $N(A)\subset\mathbb{X}$ and $R(A)\subset\mathbb{Y}$ are (vector) subspaces. Furthermore, $R(A)$ may or may not be closed while $N(A)$ is always closed.

**Definition 16.** If $A\in B(\mathbb{X},\mathbb{Y})$, then

(1) $A$ is said to be one-to-one if $N(A)=\{0\}$;
(2) $A$ is said to be onto if $R(A)=\mathbb{Y}$;

(3) $A$ is said to be invertible if it is both one-to-one and onto.

If $A$ is invertible, then there exists a unique bounded linear operator denoted $A^{-1} : \mathbb{Y} \mapsto \mathbb{X}$ called the inverse of $A$ such that

$$A^{-1}A = I_{\mathbb{X}}, \text{ and } AA^{-1} = I_{\mathbb{Y}},$$

where $I_{\mathbb{X}}$ and $I_{\mathbb{Y}}$ are the identity operators on $\mathbb{X}$ and $\mathbb{Y}$, respectively. When $\mathbb{X} = \mathbb{Y}$, the identity operator is denoted by $I$.

**Lemma 2.** *Let* $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ *is a non-Archimedean Banach space over* $\mathbb{K}$ *and let* $A \in B(\mathbb{X})$. *If* $\|A\| < 1$, *then* $I - A$ *is invertible, and*

$$(I-A)^{-1} = \sum_{t\geq 0} A^t \tag{3.2.4}$$

*with* $\|I - A\| \leq 1$.

*Proof.* First of all, let us check that the series $\sum_{t\geq 0} A^t$ is well defined. This is actually equivalent to the convergence of $A^t$ to zero in $B(\mathbb{X})$, as $t \to \infty$. Clearly, $\|A^t\| \leq \|A^{t-1}\|\|A\| \leq \|A\|^t$ for each $t \in \mathbb{N}$, and hence $\|A\|^t \to 0$ as $t \to \infty$, by $\|A\| < 1$. In view of the above, the series $\sum_{t\geq 0} A^t$ is absolutely convergent. Denote its sum by $S := \sum_{t\geq 0} A^t$. Clearly, $S \in B(\mathbb{X})$. Indeed,

$$\begin{aligned}
\|S\| &= \|\lim_{t\to\infty} \sum_{s=0}^{t} A^s\| \\
&\leq \lim_{t\to\infty} \|\sum_{s=0}^{t} A^s\| \\
&\leq \lim_{t\to\infty} \max(1, \|A\|, \|A^2\|, ...., \|A^t\|) \\
&\leq \lim_{t\to\infty} \max(1, \|A\|, ...., \|A\|^t) \\
&= 1.
\end{aligned}$$

Moreover,

$$(I-A)S = (I-A)\lim_{t\to\infty} \sum_{s=0}^{t} A^s = \lim_{t\to\infty} (I - A^{t+1}) = I.$$

Similarly, it can be easily shown that $R(I-A) = \mathbb{X}$, and hence $S = (I-A)^{-1} \in B(\mathbb{X})$ with $\|(I-A)^{-1}\| \leq 1$.

Let $(\mathbb{X}, \|\cdot\|)$ be a non-Archimedean Banach space over $\mathbb{K}$ and let $A \in B(\mathbb{X})$. The *resolvent* set of $A$ is the set of all $\lambda \in \mathbb{K}$ such that the operator $A - \lambda I$ has an inverse operator $(A - \lambda I)^{-1} \in B(\mathbb{X})$. The *spectrum* $\sigma(A)$ of $A$ is the set of all $\lambda \in \mathbb{K}$ such that $A - \lambda I$ is not invertible, that is, $\sigma(A) = \mathbb{K} - \rho(A)$.

*Example 14.* Let $\mathbb{X} = C(\mathbb{Z}_p, \mathbb{Q}_p)$ equipped with the sup norm. Consider the multiplication operator given in Example 13 (with $\gamma(x) = x$) defined by

$$M : C(\mathbb{Z}_p, \mathbb{Q}_p) \mapsto C(\mathbb{Z}_p, \mathbb{Q}_p), \quad M\phi(x) := x\,.\,\phi(x).$$

It is not hard to check that both the spectrum and the resolvent set of $M$ are respectively given by

$$\sigma(M) = \mathbb{Z}_p, \text{ and } \rho(M) = \mathbb{Q}_p - \mathbb{Z}_p.$$

A scalar $\lambda \in \mathbb{K}$ is called *eigenvalue* of an operator $A \in B(\mathbb{X})$ if there exists $0 \neq u \in \mathbb{X}$ such that $Au = \lambda u$. In this event, $u$ is called an *eigenvector* associated with the eigenvalue $\lambda$. The set of all eigenvalues is called the *point* spectrum and denoted by $\sigma_P(A)$. As in the classical setting, the following holds:

**Proposition 14.** *Let $A \in B(\mathbb{X})$. Then $\sigma_P(A) \subset \sigma(A)$.*

*Proof.* Let $\lambda \in \sigma_P(A)$ and let $0 \neq u \in \mathbb{X}$ be an eigenvector associated with $\lambda$. Clearly, $A - \lambda I$ is not one-to-one, by $(A - \lambda I)u = 0$ and $(A - \lambda I)0 = 0$ with $u \neq 0$, and hence $\lambda \in \sigma(A)$.

*Example 15.* Let $p$ be an odd prime. Let us equip $\mathbb{Q}_p \times \mathbb{Q}_p$ with the non-Archimedean norm given by $\|(x,y)\| = \max(|x|,|y|)$ for all $x,y \in \mathbb{Q}_p$. Consider the $2 \times 2$ square matrix defined by

$$T = \begin{pmatrix} a\ b \\ b\ a \end{pmatrix}, \quad a,b \in \mathbb{Q}_p - \{0\}.$$

One can easily check that:

(2) $\sigma(T) = \sigma_P(T) = \{a-b, a+b\}$;
(3) $\rho(T) = \mathbb{Q}_p - \{a-b, a+b\}$;
(3) $\|T\| = \max(|a-b|, |a+b|) = \max(|a|, |b|)$.

We will retrieve the matrix $T$ in Chapter 6 through a study of its functions.

Let $A \in B(\mathbb{X})$. Define the resolvent $R^A_\lambda$ for each $\lambda \in \rho(A)$ by

$$R^A_\lambda = (A - \lambda I)^{-1}.$$

By definition of an element $\lambda \in \rho(A)$,

$$R^A_\lambda = (A - \lambda I)^{-1} \in B(\mathbb{X}).$$

As in the classical setting, the so-called resolvent equation holds in the non-Archimedean context:

**Proposition 15.** *Let $A \in B(\mathbb{X})$. Then the resolvent equation holds, that is,*

$$R^A_\lambda - R^A_\mu = (\lambda - \mu)\,.\,R^A_\lambda R^A_\mu$$

*for all $\lambda, \mu \in \rho(A)$.*

*Proof.* Write

$$R^A_\lambda - R^A_\mu = R^A_\lambda[(A - \mu I) - (A - \lambda I)]R^A_\mu = (\lambda - \mu)R^A_\lambda\, R^A_\mu.$$

## 3.3 Bounded Linear Operators on Hilbert Spaces $\mathbb{E}_\omega$

### 3.3.1 Introduction

This section examines bounded linear operators on $\mathbb{E}_\omega$. Our first task consist of studying the decomposition of those operators as (non-Archimedean) infinite matrices. The second task, consists of taking a closer look into the crucial issue of the existence or not of the adjoint with respect to the non-Archimedean inner product given in (2.3.2).

### 3.3.2 Representation of Bounded Operators By Infinite Matrices

This subsection is mainly concerned with the decomposition of bounded linear operators in terms of infinite matrices.

For a Banach space $(\mathbb{E}, \|\cdot\|)$, let $(\mathbb{E}^*, \|\cdot\|_*)$ be its (topological) dual. For $(u,v) \in \mathbb{E} \times \mathbb{E}^*$, define the linear operator $(v \otimes u)$ by

$$\forall x \in \mathbb{E},\ (v \otimes u)(x) := v(x)u = \langle v, x \rangle u.$$

More generally, let $\mathbb{E}$ and $\mathbb{F}$ be free Banach spaces over the same field $\mathbb{K}$ with canonical orthogonal bases $(e_t)_{t\in I}$ and $(f_t)_{t\in J}$, respectively ($I, J$ being arbitrary index sets). Now if $f' \in \mathbb{E}^*$ (dual of $\mathbb{E}$), then one defines the linear functional $f' \otimes g : \mathbb{E} \mapsto \mathbb{F}$ by setting:

$$(f' \otimes g)(h) := \langle f', h \rangle g \ \text{ with} \| f' \otimes g \| = \| f' \| \| g \|.$$

In particular if $(e_t')_{t\in I}$ is the dual canonical orthogonal base for $\mathbb{E}^*$ it can be easily seen that $(e_t' \otimes f_s)_{(t,s)\in I\times J} \in B(\mathbb{E}, \mathbb{F})$, moreover

$$\|e_t' \otimes f_s\| = \frac{\|f_s\|}{\|e_t\|}.$$

Clearly, if $A \in B(\mathbb{E}, \mathbb{F})$, then it can be decomposed through the base $(f_t)_{t\in J}$ as follows: For all $t \in I$,

$$Ae_t = \sum_{s\in J} a_{st} f_s \ \text{ with } \ \lim_{s\in J} |a_{st}| \|f_s\| = 0, \ \ \forall t \in I.$$

Consequently, $A = \sum_{ts} a_{ts} e_t' \otimes f_s$. Furthermore, from the definition of the norm of $A$, i.e., $\|A\| := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$ it follows that

$$\begin{aligned} \|A\| &= \sup_t \frac{\|Ae_t\|}{\|e_t\|} \\ &= \sup_t \sup_s \frac{|a_{st}| \|f_s\|}{\|e_t\|}. \end{aligned}$$

The previous discussion can be formulated by:

**Proposition 16.** *If $A \in B(\mathbb{E},\mathbb{F})$, then it can be written in a unique fashion as a pointwise convergent series*

$$A = \sum_{ts} a_{ts} e'_t \otimes f_s, \quad t \in I, \quad \lim_s |a_{ts}| \| f_s\| = 0. \tag{3.3.1}$$

*Moreover,* $\|A\| = \sup_t \sup_s \frac{|a_{st}| \| f_s\|}{\|e_t\|}$.

*Example 16.* Suppose that $\mathbb{E} = \mathbb{F}$ and let $A = \sum_{ts} a_{ts} e'_t \otimes e_s$, where $a_{ts} = \lambda \in \mathbb{K} - \{0\}$ for all $(t,s) \in I \times J$. If $\|e_t\| = \alpha > 0$ for all $t$, then $A$ is a bounded linear operator. Moreover, $\|A\| = |\lambda|$.

If $(\mathbb{E}, \|\cdot\|)$ is a free Banach, then for every linear operator $A$ on $\mathbb{E}$, the domain $D(A)$ of $A$ is defined by

$$D(A) := \{x = (x_t) \in \mathbb{E} : \lim_t |x_t| \|Ae_t\| = 0\}. \tag{3.3.2}$$

In fact, if $A$ is a bounded linear operator, then $D(A) = \mathbb{E}$. Indeed, for each $x = \sum_t x_t e_t$ in $\mathbb{E}$,

$$|x_t| \|Ae_t\| \leq \|A\| |x_t| \|e_t\| = \|A\| |x_t| |\omega_t|^{1/2}.$$

Note that there are linear operators on $\mathbb{E}$ such that $D(A) \neq \mathbb{E}$. Those operators are called unbounded linear operators and will be studied in Chapter 3.

Throughout the next subsection, we suppose that $\mathbb{E} = \mathbb{F} = \mathbb{E}_\omega$, $I = J = \mathbb{N}$, and study the existence of the adjoint of a bounded linear operator on $\mathbb{E}_\omega$.

### 3.3.3 Existence of the Adjoint

In contrast with the classical operator theory, we will see that there exist bounded linear operators on $\mathbb{E}_\omega$, which do not have adjoint with respect to the non-Archimedean inner product given in (2.3.2). Therefore, a necessary and sufficient condition, which ensures the existence of the adjoint will be given. For more on the present setting we refer the reader to **[4]**, **[24]**, and **[25]**.

Let $\omega = (\omega_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of nonzero elements and let $\mathbb{E}_\omega$ be its corresponding non-Archimedean Hilbert space. Let $A, B$ be bounded linear operators on $\mathbb{E}_\omega$. As we have previously seen, both $A$ and $B$ can be decomposed as follows:

$$A = \sum_{t,s\in\mathbb{N}} a_{ts} e'_s \otimes e_t, \quad \forall s \in \mathbb{N}, \quad \lim_{t\to\infty} |a_{ts}| |\omega_t|^{1/2} = 0,$$

and

$$B = \sum_{t,s\in\mathbb{N}} b_{ts}\, e'_s \otimes e_t, \quad \forall s \in \mathbb{N}, \quad \lim_{t\to\infty} |b_{ts}| |\omega_t|^{1/2} = 0.$$

**Definition 17.** The linear operator $B$ given above is called the adjoint of $A$ if and only if, $\langle A\phi,\psi\rangle = \langle\phi,B\psi\rangle$, for all $\phi,\psi\in\mathbb{E}_\omega$, where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{E}_\omega$ given in (2.3.2).

*Remark 4.* (1) If the adjoint of an operator exists, then it is unique. The uniqueness of the adjoint is actually guaranteed by the fact that the inner product $\langle\cdot,\cdot\rangle$ in (2.3.2) is non-degenerate.

(2) It is sufficient to define the adjoint of $A$ using the canonical basis $(e_t)_{t\in\mathbb{N}}$ for $\mathbb{E}_\omega$. Thus $B$ is an adjoint for $A$ if and only if:

$$\langle Ae_t,e_s\rangle = \langle e_t,Be_s\rangle,\ \ \forall t,s\in\mathbb{N}. \tag{3.3.3}$$

(3) The equation (3.3.3) is equivalent to: $b_{ts} = w_t^{-1}w_s a_{st}$ for all $t,s\in\mathbb{N}$. Moreover, since $\lim_{t\to\infty}|b_{ts}||\omega_t|^{1/2} = 0,\ \ \forall s\in\mathbb{N}$, the operator $A$ has an adjoint if and only if

$$\lim_{s\to\infty}\left(\frac{|a_{ts}|}{|\omega_s|^{1/2}}\right) = 0,\ \ \forall t\in\mathbb{N}. \tag{3.3.4}$$

(4) The adjoint of an operator $A$ is denoted by $A^*$.

**Theorem 4.** *Let $A$ be the bounded linear operator given above. Then $A$ has an adjoint $A^*\in B(\mathbb{E}_\omega)$ if and only (3.3.4) holds. Under (3.3.4), the adjoint $A^*$ of $A$ can be uniquely expressed by*

$$A^* = \sum_{(t,s)\in\mathbb{N}\times\mathbb{N}} w_t^{-1}\omega_s\, a_{st}\, e_s'\otimes e_t. \tag{3.3.5}$$

*Proof.* Write $A^* = \sum_{t\in\mathbb{N}}\sum_{s\in\mathbb{N}} b_{st}\left(e_t'\otimes e_s\right)$, then $A^*$ is the adjoint of $A$ if and only if $\langle Ae_s,e_t\rangle = \langle e_s,A^*e_t\rangle$. In other terms:

$$\left\langle \sum_{k\in\mathbb{N}} a_{ks}e_k, e_t\right\rangle = a_{ts}\omega_t = \left\langle e_s, \sum_{k\in\mathbb{N}} b_{kt}e_k\right\rangle = b_{st}\omega_s,\quad \forall s,t\in\mathbb{N}.$$

This is equivalent to $b_{st} = \omega_s^{-1}\omega_t a_{ts}$ for all $s,t\in\mathbb{N}$. Moreover, for all $t$,

$$\lim_{s\to\infty}|b_{st}|\,|\omega_s|^{1/2} = 0.$$

This is equivalent to $\lim_{t\to\infty}\frac{|a_{st}|}{|\omega_t|^{1/2}} = 0$, for all $s\in\mathbb{N}$.

The collection of all bounded linear operators on $\mathbb{E}_\omega$ whose adjoint do exist with respect to the non-Archimedean inner product $\langle\cdot,\cdot\rangle$ given in (2.3.2), is denoted by $B_0(\mathbb{E}_\omega)$. Namely,

$$B_0(\mathbb{E}_\omega) := \{\sum_{t,s\in\mathbb{N}} a_{ts}\, e_s'\otimes e_t\in B(\mathbb{E}_\omega):\ \lim_{s\to\infty}\frac{|a_{ts}|}{|\omega_s|^{1/2}} = 0,\ \forall t\in\mathbb{N}\}.$$

*Remark 5.* Note that $B_0(\mathbb{E}_\omega)$ is stable under the operation of taking an adjoint. Namely, for all $A, B \in B_0(\mathbb{E}_\omega)$ and $\lambda \in \mathbb{K}$, the following hold:

(1) $(A+B)^* = A^* + B^*$;
(2) $(AB)^* = B^* A^*$;
(3) $(\lambda A)^* = \lambda A^*$;
(4) $(A^*)^* = A$;
(5) $\|A\| = \|A^*\|$.

### 3.3.4 Examples of Bounded Operators with no Adjoint

*Example 17.* Let $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$. Suppose that $\omega_t = 1 + p^{1+t}$ for each $t \in \mathbb{N}$. Define the linear operator $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ by:

$$A = \sum_{t,s} a_{ts}(e'_s \otimes e_t),$$

where $a_{ts} = p^{1+t}$ for all $t, s \in \mathbb{N}$.

**Proposition 17.** *The operator A defined above is bounded and does not have an adjoint.*

*Proof.* The operator $A = \sum_{t,s} p^{1+t}(e'_s \otimes e_t)$ is well-defined. Indeed, $\forall s$, $\lim_{t\to\infty} |a_{ts}|.|\omega_t|^{1/2} = \lim_{t\to\infty} p^{-(1+t)} = 0$. Furthermore,

$$\|A\| = \sup_{t,s\in\mathbb{N}} \frac{|a_{ts}|.|\omega_t|^{1/2}}{|\omega_s|^{1/2}} = \sup_{t,s\in\mathbb{N}} p^{-(1+t)} = \frac{1}{p} < \infty.$$

It remains to prove that $\forall t$, $\lim_{s\to\infty} \frac{|a_{ts}|}{|\omega_s|^{1/2}} \neq 0$. Indeed,

$$\forall t, \ \lim_{s\to\infty} \frac{|a_{ts}|}{|\omega_s|^{1/2}} = \lim_{s\to\infty} p^{-(1+t)} = p^{-(1+t)} \neq 0,$$

hence $A$ does not have an adjoint.

Example 17 is a particular case of the next example.

*Example 18.* Let $(\mathbb{K}, |\cdot|)$ be a non-Archimedean valued field. Suppose that $|\omega_t| = 1$ for each $t \in \mathbb{N}$. Define the linear operator $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ by: $A = \sum_{t,s} a_{ts}\,(e'_s \otimes e_t)$, where $a_{ts} = \vartheta_t$ for all $t, s \in \mathbb{N}$ with $|\vartheta_t| \neq 0$ for each $t \in \mathbb{N}$ and $\lim_{t\to\infty} |\vartheta_t| = 0$.

**Proposition 18.** *The operator A defined above is bounded and does not have an adjoint.*

*Proof.* The proof is similar to that of Proposition 17, and therefore left to the reader.

## 3.4 Perturbation of Bases

Let an orthogonal base $(h_s)_{s\in\mathbb{N}}$ be given and consider a sequence of vectors $(f_s)_{s\in\mathbb{N}}$, not necessarily an orthogonal base, in $\mathbb{E}_\omega$ such that the difference $f_s - h_s$ is small in a certain sense. We study some sufficient conditions under which the family $(f_s)_{s\in\mathbb{N}}$ is a base of $\mathbb{E}_\omega$.

**Theorem 5.** *Let $(h_s)_{s\in N}$ be an orthogonal base and let $(f_s)_{s\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{E}_\omega$ satisfying the following condition: There exists $\alpha \in (0,1)$ such that for every $x = \sum_{s\in\mathbb{N}} x_s h_s \in \mathbb{E}_\omega$,*

$$\sup_{s\in\mathbb{N}} |x_s|\,\|f_s - h_s\| \le \alpha\,.\,\sup_{s\in\mathbb{N}} |x_s|\,\|h_s\| = \alpha\,.\,\|x\|\,.$$

*Then $(f_s)_{s\in\mathbb{N}}$ is an orthogonal base. Moreover, for any $y = \sum_{s\in\mathbb{N}} y_s f_s \in \mathbb{E}_\omega$, $\|y\| = \sup_{s\in\mathbb{N}} (|y_s|\,.\,\|f_s\|)$.*

*Proof.* We first observe that, if $x = h_s$, then, the condition implies that for each $s$, $\|f_s - h_s\| \le \alpha\|h_s\| < \|h_s\|$, hence $\|f_s\| = \|h_s\|$.

Next, for any $x = \sum_{s\in\mathbb{N}} x_s h_s \in \mathbb{E}_\omega$, $|x_s|\,\|f_s\| = |x_s|\,\|h_s\|$, that is, $\lim_{s\to\infty} (|x_s|\,.\,\|f_s\|) = 0$. Therefore, the operator defined by

$$Ax = \sum_{s\in\mathbb{N}} x_s f_s$$

is well-defined and satisfies $Ah_s = f_s$. Moreover

$$\begin{aligned}\|x - Ax\| &= \left\|\sum_{s\in\mathbb{N}} x_s (h_s - f_s)\right\| \\ &\le \sup_{s\in\mathbb{N}} (|x_s|\,.\,\|h_s - f_s\|) \\ &\le \alpha\,.\,\sup_{s\in\mathbb{N}} (|x_s|\,.\,\|h_s\|) \\ &= \alpha\,.\,\|x\|\,.\end{aligned}$$

It follows that $\|I - A\| \le \alpha < 1$, and hence $A$ is invertible, by Theorem 2.

It remains to show that $A$ is isometric. In view of the above, observe that the inequalities: $\|x - Ax\| \le \alpha\|x\| < \|x\|$ imply that $\|Ax\| = \|x\|$ for any $x \in \mathbb{E}_\omega$, hence, $A$ is isometric.

Consequently, $(f_s)_{s\in\mathbb{N}}$ is an orthogonal base. Moreover, for any $y = \sum_{s\in\mathbb{N}} y_s f_s \in \mathbb{E}_\omega$

$$\|y\| = \left\|A(\sum_{s\in\mathbb{N}} y_s h_s)\right\| = \left\|\sum_{s\in\mathbb{N}} y_s h_s\right\| = \sup_{s\in\mathbb{N}} (|y_s|\,.\,\|h_s\|).$$

Among other things, the following is an immediate consequences of Theorem 5.

**Corollary 2.** *Let $(e_s)_{s\in\mathbb{N}}$ be the canonical base for $\mathbb{E}_\omega$ and let $(f_s)_{s\in\mathbb{N}} \subset \mathbb{E}_\omega$ be a family of vectors such that*

$$\sup_{t\in\mathbb{N}}\left(\frac{\|e_t - f_t\|}{\|e_t\|}\right) < 1. \tag{3.4.1}$$

*Then* $(f_s)_{s\in\mathbb{N}}$ *is also an orthogonal base for* $\mathbb{E}_\omega$.

**Theorem 6.** *Let* $(h_s)_{s\in\mathbb{N}}$ *be an orthogonal base,* $C \in B(\mathbb{E}_\omega)$ *invertible such that* $\left\|C^{-1}\right\| = \|C\|^{-1}$. *Suppose that* $(f_s)_{s\in\mathbb{N}}$ *is a sequence of vectors in* $\mathbb{E}_\omega$ *satisfying the following condition*

$$\sup_{s\in\mathbb{N}}\frac{\|f_s - Ch_s\|}{\|h_s\|} < \|C\|,$$

*then* $(f_s)_{s\in\mathbb{N}}$ *is an orthogonal base.*

*Proof.* We first observe that for any $s$, $\|f_s\| \le \|C\|\,\|h_s\|$. For any $x = \sum_{s\in\mathbb{N}} x_s h_s \in \mathbb{E}_\omega$ :

$$\begin{aligned}\lim_{s\to\infty}|x_s|\,\|f_s\| &= \lim_{s\to\infty}|x_s|\,\|h_s\|\,\frac{\|f_s\|}{\|h_s\|}\\ &\le \|C\| \;.\; \lim_{s\to\infty}|x_s|\,\|h_s\|\\ &= 0.\end{aligned}$$

Therefore if we put $Ax = \sum_{s\in\mathbb{N}} x_s f_s$, then, $A$ is a well-defined operator satisfying $Ah_s = f_s$.

The second condition of the theorem implies that if we put $B = C - A$, then

$$\|B\| = \|C - A\| = \|A - C\| < \|C\|,$$

from which we deduce that $\|A\| = \|C\|$.

Now

$$\left\|BC^{-1}\right\| \le \|B\|\left\|C^{-1}\right\| < \|C\|\left\|C^{-1}\right\| = 1,$$

by assumption. Therefore, the operator $AC^{-1}$ is such that

$$\left\|1 - AC^{-1}\right\| = \left\|BC^{-1}\right\| < 1.$$

We can apply Theorem 2 to $AC^{-1}$ and find that it is invertible. Since $C$ is also invertible, it follows that $A$ is invertible. Moreover, it is not difficult to show that $A$ is isometric, and hence, $(f_s)_{s\in\mathbb{N}}$ is an orthogonal base for $\mathbb{E}_\omega$.

The next corollary is a generalization of Corollary 2.

**Corollary 3.** *Let* $(h_s)_{s\in\mathbb{N}} \subset \mathbb{E}_\omega$ *be an orthogonal basis,* $(f_s)_{s\in\mathbb{N}} \subset \mathbb{E}_\omega$ *a sequence of vectors and* $\zeta$ *a non-zero element of* $\mathbb{K}$ *satisfying:*

$$\sup_{s\in\mathbb{N}}\frac{\|f_s - \zeta h_s\|}{\|h_s\|} < |\zeta|,$$

*then,* $(f_s)_{s\in\mathbb{N}}$ *is an orthogonal base for* $\mathbb{E}_\omega$.

*Proof.* In Theorem 6, we take the matrix $C$ to be the diagonal matrix $C = \sum_{t,s\in\mathbb{N}} c_{ts}\left(h_s'\otimes h_t\right)$ with $c_{ts} = 0$ if $t \neq s$ and $c_{tt} = \zeta$ for all $t \geq 0$. It is clear that for any $s$, $Ch_s = \zeta h_s$, $C$ is invertible, $C^{-1}h_s = \zeta^{-1}h_s$, $\|C\| = |\zeta|$ and $\left\|C^{-1}\right\| = |\zeta|^{-1}$. Moreover

$$\begin{aligned}\sup_{s\in\mathbb{N}} \frac{\|f_s - Ch_s\|}{\|h_s\|} &= \sup_{s\in\mathbb{N}} \frac{\|f_s - \zeta h_s\|}{\|h_s\|}\\ &< |\zeta|\\ &= \|C\|.\end{aligned}$$

**Corollary 4.** *Let $(h_s)_{s\in\mathbb{N}}$ be an orthogonal base for $\mathbb{E}_\omega$. If $(g_s)_{s\in\mathbb{N}}$ is a sequence of vectors of $\mathbb{E}_\omega$ satisfying: $\lim_{s\to\infty} \frac{|\langle h_k, g_s\rangle|}{|\omega_s|}\|g_s\| = 0$ for each $k \in \mathbb{N}$, and that,*

$$\sup_{s\in\mathbb{N}}\left(\frac{\|h_s - Sh_s\|}{|\omega_s|^{1/2}}\right) < 1,$$

*where $Sh_k = \sum_{s\in\mathbb{N}} \frac{\langle h_k, g_s\rangle}{\omega_s} g_s$ for each $k \in \mathbb{N}$, then $(Sh_s)_{s\in\mathbb{N}}$ is an orthogonal base of $\mathbb{E}_\omega$.*

*Remark 6.* (1) Note that $S$ defined above is a linear operator on $\mathbb{E}_\omega$.
(2) The assumption, $\lim_{s\to\infty} \frac{|\langle h_k, g_s\rangle|}{|\omega_s|}\|g_s\| = 0$ implies that

$$Sh_k = \sum_{s\in\mathbb{N}} \frac{\langle h_k, g_s\rangle}{\omega_s} g_s, \quad \forall k \in \mathbb{N}$$

is well-defined.
(3) The operator $S$ is isometric, by using $\sup_{s\in\mathbb{N}}\left(\frac{\|h_s - Sh_s\|}{|\omega_s|^{1/2}}\right) < 1$.

*Proof.* It suffices to put $C = I$, and $f_s = Sh_s$ in Theorem 6.

### 3.4.1 Example

We illustrate Corollary 3 with the following example: Let $p$ be a prime, $\mathbb{K} = \mathbb{Q}$, $\omega_s = p^s$, $|\omega_s| = \frac{1}{p^s}$.

As an orthogonal basis for $\mathbb{E}_\omega$ we use the canonical orthogonal basis $(e_s)_{s\in\mathbb{N}}$ and recall that $\|e_s\| = |\omega_s|^{1/2} = \frac{1}{p^{s/2}}$. Let $\varsigma$ be such that $|\varsigma| = 1$ and let

$$g_s = (u - \varsigma)\, e_s + p^{1+s} \sum_{t\in\mathbb{N}, t\neq s} e_t, \quad \forall s \in \mathbb{N}.$$

We choose $u$ such that $|u| = 1$ and $|u - \varsigma| = \frac{1}{p^{1+s}}$. To achieve this choice of $u$ we do the following: $\varsigma$ is a $p$-adic unit which can be written in $\mathbb{K}$ as

$$\varsigma = a_0 + a_1 p + a_2 p^2 + .... + a_s p^s + a_{s+1} p^{s+1} + ....,$$

where $1 \leq a_0 \leq p-1$ and for $k \neq 0,\ 0 \leq a_k \leq p-1$.

Setting

$$u = a_0 + a_1 p + a_2 p^2 + .... + a_s p^s$$

it follows that $|u| = 1$ and

$$u - \varsigma = -(a_{s+1} p^{s+1} + ....),$$

and hence $|u - \varsigma| = \dfrac{1}{p^{1+s}}$.

Clearly,

$$\begin{aligned} \|g_s\| &= \max\left(|u-\varsigma|\,|\omega_s|^{1/2}, \frac{1}{p^{1+s}} \sup_{i\neq s} |\omega_i|^{1/2}\right) \\ &= \max\left(\frac{1}{p^{1+s+s/2}}, \sup_{i\neq s} \frac{1}{p^{1+s+i/2}}\right) \\ &= \max\left(\frac{1}{p^{1+s+s/2}}, \frac{1}{p^{1+s}}\right) \\ &= \frac{1}{p^{1+s}} \text{ (even if } s = 0). \end{aligned}$$

Thus $\dfrac{\|g_s\|}{|\omega_s|^{1/2}} = \dfrac{1}{p^{1+s/2}}$, and

$$\sup_s \frac{\|g_s\|}{|\omega_s|^{1/2}} = \sup_s \frac{1}{p^{1+s/2}} = \frac{1}{p} < |\varsigma| = 1.$$

In summary, if we let $f_s = g_s + \zeta e_s = u e_s + p^{1+s} \sum_{i \neq s} e_i$, then

$$\sup_{s\in\mathbb{N}} \frac{\|f_s - \zeta e_s\|}{\|e_s\|} = \sup_{s\in\mathbb{N}} \frac{\|g_s\|}{|\omega_s|^{1/2}} < |\zeta|.$$

## 3.5 Hilbert-Schmidt Operators

### 3.5.1 Basic Definitions

As in the classical setting, Hilbert-Schmidt operators in non-Archimedean Hilbert spaces are bounded linear operators satisfying a convergence property. Surprisingly, such convergence depends on the base.

**Definition 18.** A bounded linear operator $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ is a $p$-adic Hilbert-Schmidt operator if:

$$Q(A) := \left[\sum_{t\in\mathbb{N}}\left(\frac{\|Ae_t\|}{\|e_t\|}\right)^2\right]^{1/2} < \infty,$$

where $(e_t)_{t\in\mathbb{N}}$ is the canonical orthogonal base for $\mathbb{E}_\omega$. We denote by $B_2(\mathbb{E}_\omega)$ the collection of all Hilbert-Schmidt operators on $\mathbb{E}_\omega$.

*Remark 7.* (i) If $\|e_t\| = 1$ for all $t \in \mathbb{N}$, i.e., if $(e_t)_{t\in\mathbb{N}}$ is an orthonormal base, we retrieve the definition of a Hilbert-Schmidt operator as in the classical context.

*Remark 8.* In contrast with the classical setting, the definition of a Hilbert-Schmidt operator depends on the base. To find out, let $p$ be an odd prime and set $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$. Then $|\frac{1}{2}| = |2| = 1$. Let $\omega_i = 1$ for $i = 0,1,2,...$ and consider the corresponding $p$-adic Hilbert space $\mathbb{E}_\omega$.

Let $(e_i)_{i\in\mathbb{N}}$ be the canonical base for $\mathbb{E}_\omega$ and let $f_i = e_i$ for $i > 1$ and $f_0 = e_0 + e_1$, and $f_1 = e_0 - e_1$. It can be shown that $(f_i)_{i\in\mathbb{N}}$ is an orthogonal base for $\mathbb{E}_\omega$.

Let $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ be the projection onto the one-dimensional space spanned by $e_0$:

$$Ae_0 = e_0, \;\; Ae_i = 0 \;\text{ for all }\; i \geq 1.$$

It can be easily shown that $A$ is a Hilbert-Schmidt operator and that

$$1 = \sum_{s\in\mathbb{N}}\frac{\|Ae_s\|^2}{|\omega_s|} \neq \sum_{s\in\mathbb{N}}\frac{\|Af_s\|^2}{\|f_s\|^2} = 2.$$

**Definition 19.** Let $A$ be a Hilbert-Schmidt operator on $\mathbb{E}_\omega$. The real number

$$Q(A) := \left[\sum_{s\in\mathbb{N}}\left(\frac{\|Ae_s\|}{\|e_s\|}\right)^2\right]^{1/2}$$

is then well-defined and is called the Hilbert-Schmidt norm of $A$.

**Theorem 7.** *Let $A \in B_2(\mathbb{E}_\omega)$, then the following statements hold true:*

*(1) $A$ has an adjoint denoted $A^*$;*
*(2) $Q(A) = Q(A^*)$ and $A^* \in B_2(\mathbb{E}_\omega)$;*
*(3) $\|A\| \leq (Q(A))$.*

*Remark 9.* In view of the above, $B_2(\mathbb{E}_\omega) \subset B_0(\mathbb{E}_\omega)$.

*Proof.* Let $A = \sum_{t,s} a_{ts}\left(e'_s \otimes e_t\right)$ be a Hilbert-Schmidt operator. We first show that the boundedness is a direct consequence of the convergence property. The convergence of the real series yields:

$$\lim_{s\to\infty}\frac{\|Ae_s\|^2}{|\omega_s|} = 0,$$

and therefore, the sequence $\left(\frac{\|Ae_s\|^2}{|\omega_s|}\right)_{s\in\mathbb{N}}$ is bounded. But since

$$\|A\| = \sup_s \frac{\|Ae_s\|}{\|e_s\|} = \sup_s \frac{\|Ae_s\|}{|\omega_s|^{1/2}},$$

hence $\|A\| < \infty$.

(1) We need to prove that for all $t$, $\lim_{s\to\infty} \frac{|a_{ts}|}{|\omega_s|^{1/2}} = 0$. Again, because of the convergence of the real series, $\lim_{s\to\infty}(\frac{\|Ae_s\|^2}{|\omega_s|}) = 0$.

Since

$$\|Ae_s\| = \|\sum_t a_{ts}e_t\| = \sup_t |a_{ts}||\omega_t|^{1/2},$$

it follows that

$$\forall t, \quad \lim_{s\to\infty} \frac{|a_{ts}|}{|\omega_s|^{1/2}} = 0.$$

(2) If $A^* = \sum_{t,s} a^*_{ts}\left(e'_s \otimes e_t\right)$ with $a^*_{ts} = \omega_t^{-1}\omega_s a_{st}$, then:

$$\begin{aligned} Q^2(A^*) &= \sum_s \frac{\|A^*(e_s)\|^2}{|\omega_s|} \\ &= \sum_s \frac{\sup_t |\omega_t|^{-2}|\omega_s|^2|a_{st}|^2|\omega_t|}{|\omega_s|} \\ &= \sum_s \sup_t \frac{|a_{st}|^2|\omega_s|}{|\omega_t|}. \end{aligned}$$

On the other hand, $Q^2(A) = \sum_s \sup_t \frac{|a_{ts}|^2|\omega_t|}{|\omega_s|}$ but if we consider the real matrix $C = (c_{ts})$ where $c_{ts} = \frac{|a_{ts}|^2|\omega_t|}{|\omega_s|}$ then its transpose is $C^t = (c_{st})$ with $c_{st} = \frac{|a_{st}|^2|\omega_s|}{|\omega_t|}$. Moreover, all the entries in column $s$ of $C$ are the same as all the entries in row $s$ of $C^t$, therefore

$$\sup_t \frac{|a_{ts}|^2|\omega_t|}{|\omega_s|} = \sup_t \frac{|a_{st}|^2|\omega_s|}{|\omega_t|}.$$

Thus, $Q^2(A^*) = Q^2(A) < \infty$. Consequently, $A^* \in B_2(\mathbb{E}_\omega)$.

(3) We have

$$\begin{aligned} Q^2(A) &= \sum_s \sup_t \frac{|a_{ts}|^2|\omega_t|}{|\omega_s|} \\ &\geq \sup_s\left(\sup_t \frac{|a_{ts}|^2|\omega_t|}{|\omega_s|}\right) \\ &= \|A\|^2. \end{aligned}$$

This completes the proof.

**Theorem 8.** *The following statements hold true:*

*(1) let $A,B \in B_0(\mathbb{E}_\omega)$ and suppose that $A \in B_2(\mathbb{E}_\omega)$, then both AB and BA are in $B_2(\mathbb{E}_\omega)$.*
*(2) let $A,B \in B_2(\mathbb{E}_\omega)$, then $AB \in B_2(\mathbb{E}_\omega)$ and*

$$Q(AB) \leq Q(A)\,.\,Q(B).$$

*Remark 10.* (a) From (1) above it follows that $B_2(\mathbb{E}_\omega)$ is a two-sided ideal in $B_0(\mathbb{E}_\omega)$;
(b) From (2) above, we have that $(B_2(\mathbb{E}_\omega), Q(\cdot))$ is a normed algebra.

*Proof.* (1) Observe that $\|BA\,(e_s)\| \leq \|B\|\,\|A\,(e_s)\|$, hence

$$\frac{\|BA\,(e_s)\|^2}{|\omega_s|} \leq \|B\|^2\,.\,\frac{\|A(e_s)\|^2}{|\omega_s|}.$$

Since $B \in B(\mathbb{E}_\omega)$ and $A \in B_2(E_\omega)$ it follows that $BA \in B_2(\mathbb{E}_\omega)$.

Now by Theorem 7, $A^*$ exists and is in $B_2(\mathbb{E}_\omega)$, hence, by the first part, $B^*A^* \in B_2(\mathbb{E}_\omega)$. Again by Theorem 7, $AB = (B^*A^*)^* \in B_2(\mathbb{E}_\omega)$.

(2) This is straightforward, and hence left to the reader.

The following is now clear.

**Theorem 9.** *Let $A \in B_0(\mathbb{E}_\omega)$, then the following are equivalent*

*(1) $A \in B_2(\mathbb{E}_\omega)$;*
*(2) for all $U,V \in B_0(\mathbb{E}_\omega), UAV \in B_2(\mathbb{E}_\omega)$.*

*Proof.* To see that $(2) \Rightarrow (1)$ choose $U = V = I$.

In the space $B_2(\mathbb{E}_\omega)$ we introduce a form, namely: For all $A,B \in B_2(\mathbb{E}_\omega)$,

$$\langle A,B\rangle := \sum_s \frac{\langle Ae_s, Be_s\rangle}{\omega_s}. \tag{3.5.1}$$

This defines a symmetric, bilinear and non-degenerate form on $B_2(\mathbb{E}_\omega)$. The relationship between the bilinear form defined above and the Hilbert-Schmidt norm is given by the Cauchy-Schwarz inequality (see Theorem 10 below).

**Theorem 10.** *If $A,B \in B_2(\mathbb{E}_\omega)$, then $|\langle A,B\rangle| \leq Q(A)\,.\,Q(B)$.*

*Proof.* We first give a proof along a classical line:

$$\begin{aligned} |< A,B >| &= \left|\sum_s \frac{\langle Ae_s, Be_s\rangle}{\omega_s}\right| \\ &\leq \sum_s \frac{|\langle Ae_s, Be_s\rangle|}{|\omega_s|} \\ &\leq \sum_s \frac{\|Ae_s\|\,\|Be_s\|}{|\omega_s|}, \end{aligned}$$

by the Cauchy-Schwarz inequality for the bilinear form on $\mathbb{E}_\omega$.

Now we consider the sequences $u = (u_s)_s$ with $u_s = \dfrac{\|Ae_s\|}{|\omega_s|^{1/2}}$ and $v = (v_s)_s$ with $v_s = \dfrac{\|Be_s\|}{|\omega_s|^{1/2}}$. Since both $A$ and $B$ are in $B_2(\mathbb{E}_\omega)$, then both $u$ and $v$ are in $l^2(\mathbb{N})$. By Holder's inequality $uv = (u_s v_s)_s \in l^1(\mathbb{N})$ and

$$\begin{aligned} |uv|_1 &= \sum_s \frac{\|Ae_s\| \,.\, \|Be_s\|}{|\omega_s|} \\ &\leq \left( \sum_s \frac{\|Ae_s\|^2}{|\omega_s|} \right)^{1/2} \left( \sum_s \frac{\|Be_s\|^2}{|\omega_s|} \right)^{1/2} \\ &= Q(A) \,.\, Q(B). \end{aligned}$$

Combining, one obtains $|\langle A, B\rangle| \leq Q(A) \,.\, Q(B)$.

Next one gives another proof along an ultrametric line: first, write $A = \sum_{t,s} a_{ts} \left( e'_s \otimes e_t \right)$ and $B = \sum_{t,s} b_{ts} \left( e'_s \otimes e_t \right)$ :

$$\begin{aligned} |\langle A, B\rangle| &= \left| \sum_s \frac{\langle Ae_s, Be_s\rangle}{\omega_s} \right| \\ &\leq \sup_s \frac{\|Ae_s\| \, \|Be_s\|}{|\omega_s|} \\ &\leq \left( \sup_s \frac{\|Ae_s\|}{|\omega_s|^{1/2}} \right) . \left( \sup_s \frac{\|Be_s\|}{|\omega_s|^{1/2}} \right) \\ &= \sup_s \frac{\|Ae_s\|}{\|e_s\|} . \sup_s \frac{\|Be_s\|}{\|e_s\|} \\ &= \|A\| \, \|B\| \\ &\leq Q(A) \,.\, Q(B), \end{aligned}$$

by [Theorem 7, (3)].

Here are other properties that are reminiscent of the classical case.

**Proposition 19.** *Let* $A = \sum_{ts} a_{ts} \left( e'_s \otimes e_t \right)$ *and suppose that the expression*

$$\sum_{t,s \in \mathbb{N}} \frac{|a_{ts}|^2 \, |\omega_t|}{|\omega_s|} = C^2 < \infty.$$

*Then* $A \in B_2(\mathbb{E}_\omega)$ *and* $Q(A) \leq C$.

*Proof.* We observe that $\forall s,\ \sup_t |a_{ts}|^2 \, |\omega_t| \leq \sum_t |a_{ts}|^2 \, |\omega_t|$ . Now

$$\begin{aligned}\sum_s \frac{\|Ae_s\|^2}{|\omega_s|} &= \sum_{s\in\mathbb{N}} \frac{1}{|\omega_s|} \sup_t |a_{ts}|^2 |\omega_t| \\ &\leq \sum_{s\in\mathbb{N}} \frac{1}{|\omega_s|} \left( \sum_t |a_{ts}|^2 |\omega_t| \right) \\ &= \sum_{t,s\in\mathbb{N}} \frac{|a_{ts}|^2 |\omega_t|}{|\omega_s|} \\ &= C^2.\end{aligned}$$

Therefore, $Q^2(A) \leq C^2$.

*Remark 11.* In contrast with the classical case, strict inequality may occur in Proposition 19. Indeed, let $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$, $\omega_t = p^{-t}$, $a_{ts} = p^t$, hence $|\omega_t| = p^t$, $|a_{ts}| = p^{-t}$.

(1) $\forall s,\ \lim_t |a_{ts}|\,|\omega_t|^{1/2} = \lim_t \frac{1}{p^{t/2}} = 0$.

(2)

$$\begin{aligned}\sum_{t,s} \frac{|a_{ts}|^2 |\omega_t|}{|\omega_s|} &= \sum_{t,s} \frac{p^t}{p^{2t+s}} \\ &= \sum_{t,s} \frac{1}{p^{t+s}} \\ &= C^2 > 1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_s \frac{\|Ae_s\|^2}{|\omega_s|_p} &= \sum_s \frac{1}{|\omega_s|} \sup_t |a_{ts}|^2 |\omega_t| \\ &= \sum_s \frac{1}{|\omega_s|} \sup_t \frac{1}{p^t} \\ &= \sum_s \frac{1}{|\omega_s|} \\ &= \frac{1}{1-\frac{1}{p}} \\ &= \frac{p}{p-1}.\end{aligned}$$

But since $\frac{p}{p-1} > 1$ it follows that $\frac{p}{p-1} < \frac{p^2}{(p-1)^2}$.

**Proposition 20.** *Let* $A = \sum_{t,s} a_{ts} \left(e'_s \otimes e_t\right) \in B(\mathbb{E}_\omega)$, *furthermore, suppose*

*(1)* $|\omega_s| > 1$, *for all* $s \in \mathbb{N}$;
*(2)* $\sup_s |\omega_s| = M < \infty$;

*(3)* $\sum_s (\sup_t |a_{ts}|)^2 < \infty$.

*Then* $A \in B_2(\mathbb{E}_\omega)$.

*Proof.* We simply observe that

$$\frac{\|Ae_s\|^2}{|\omega_s|} = \frac{\left(\sup_t |a_{ts}|\,|\omega_t|^{1/2}\right)^2}{|\omega_s|} \leq M\left(\sup_t |a_{ts}|\right)^2.$$

*Remark 12.* Note that Proposition 20 above is reminiscent of the properties of operators, which are of the ”Kernel” type.

### 3.5.2 Further Properties of Hilbert-Schmidt Operators

We now explore properties of Non-Archimedean Hilbert-Schmidt operators which are suggested by their classical counterparts. Set

$$c_0(\mathbb{E}_\omega^*) := \{(A_t)_{t\in\mathbb{N}} \in (\mathbb{E}_\omega^*)^{\mathbb{N}} : \lim_{t\to\infty} \|A_t\||\omega_t|^{1/2} = 0\}.$$

Clearly $c_0(\mathbb{E}_\omega^*)$ is a non-Archimedean Banach space over $\mathbb{K}$ when endowed the norm

$$\|(A_t)_{t\in\mathbb{N}}\| = \sup_{t\in\mathbb{N}} \|A_t\||\omega_t|^{1/2}.$$

Also, let

$$F_\omega := \{(x_t)_{t\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sup_{t\in\mathbb{N}} \frac{|x_t|}{|\omega_t|^{1/2}} < \infty\}.$$

In the same way $F_\omega$ is a non-Archimedean Banach space over $\mathbb{K}$ with the norm

$$\|(x_t)_t\| = \sup_{t\in\mathbb{N}} \frac{|x_t|}{|\omega_t|^{1/2}}.$$

In the context of the present situation we refer to $c_0(\mathbb{E}_\omega^*)$ as the set of all sequences of elements in $\mathbb{E}_\omega^*$ which converge to 0, and to $F_\omega$ as the set of bounded sequences of elements of $\mathbb{K}$.

**Proposition 21.** *As Banach spaces over* $\mathbb{K}$, $\mathbb{E}_\omega^*$ *is isomorphic to* $F_\omega$.

*Proof.* As in [65], the mapping $A \to (Ae_s)_s$ gives the desired isomorphism.

### 3.5.3 Completely Continuous Operators

**Definition 20.** An operator $A \in B(\mathbb{E}_\omega)$ is *completely continuous* if it is the (uniform) limit in $B(\mathbb{E}_\omega)$ of a sequence of operators of *finite rank.* We denote by $C(\mathbb{E}_\omega)$ the subspace of all completely continuous operators on $\mathbb{E}_\omega$.

**Definition 21.** For any operator $A \in B(\mathbb{E}_\omega)$, we define $A_s$, the $s-$*component* of $A$ by the formula:

$$x \in \mathbb{E}_\omega,\ Ax = (A_s x)_{s\in\mathbb{N}}\,.$$

It is clear that $A_s \in \mathbb{E}_\omega^*$.

**Proposition 22.** *The mapping $A \to (A_s)_s$ is an isomorphism of the Banach space $C(\mathbb{E}_\omega)$ onto the Banach space $c_0\left(\mathbb{E}_\omega^*\right)$.*

*Proof.* The proof is similar to that of [65, Proposition 4].

**Theorem 11.** *Every Hilbert-Schmidt operator is completely continuous, i.e., $B_2(\mathbb{E}_\omega) \subset C(\mathbb{E}_\omega)$.*

*Proof.* Let $A$ be a Hilbert-Schmidt operator, then, by Theorem 7, the adjoint $A^*$ does exist and it is also a Hilbert-Schmidt operator, hence

$$\sum_s \frac{||A^* e_s||^2}{|\omega_s|} < \infty.$$

It follows that $\lim_s \dfrac{||A^* e_s||}{|\omega_s|^{1/2}} = 0$.

Let $A = \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij}(e'_j \otimes e_i)$ be the standard representation of $A$ and $A_s$ the $s$-component of $A$. It is enough to show that

$$\lim_s ||A_s|||\omega_s|^{1/2} = 0.$$

Observe

$$\begin{aligned} ||A_s|| &= \sup_{x\neq 0} \frac{|A_s x|}{||x||} \\ &= \sup_i \frac{|A_s e_i|}{||e_i||} \\ &= \sup_i \frac{|a_{si}|}{|\omega_i|^{1/2}}. \end{aligned}$$

Hence

$$||A_s||\,|\omega_s|^{1/2} = \sup_i \frac{|a_{si}|\,|\omega_s|^{1/2}}{|\omega_i|^{1/2}} = \frac{||A^* e_s||}{|\omega_s|^{1/2}}.$$

Therefore $\lim_s ||A_s||\,|\omega_s|^{1/2} = 0$.

### 3.5.4 Trace

One important notion in the classical theory is that of trace. We now define the trace of an operator in the non-Archimedean context.

**Definition 22.** *[12]* For $A \in B_0(\mathbb{E}_\omega)$, we define the trace of $A$ to be

$$\operatorname{tr}(A) := \sum_{s\in\mathbb{N}} \frac{\langle Ae_s, e_s\rangle}{\langle e_s, e_s\rangle} \tag{3.5.2}$$

if the sum converges in $\mathbb{K}$ and where $(e_s)_{s\in\mathbb{N}}$ is the canonical orthogonal base for $\mathbb{E}_\omega$.

We denote by $B_1(\mathbb{E}_\omega)$ the collection of operators which have traces[1].

*Remark 13.* (1) The convergence in $\mathbb{K}$ of the series in (3.5.2) is equivalent to

$$\lim_{s\to\infty} \left| \frac{\langle Ae_s, e_s\rangle}{\langle e_s, e_s\rangle} \right| = 0.$$

(2) Since $A \in B_0(E_\omega)$, its adjoint $A^*$ exists and $\operatorname{tr}(A) = \operatorname{tr}(A^*)$.

*Remark 14.* Let $A$ be a Hilbert-Schmidt operator. Does the definition of $\operatorname{tr}(A)$ dependent of the canonical orthogonal base $(e_s)_{s\in\mathbb{N}}$?

### 3.5.5 Examples

Throughout this subsection, we suppose that $\mathbb{K}$ a complete non-Archimedean field, $\omega = (\omega_i)_{i\in\mathbb{N}}$ is a sequence of nonzero elements in $\mathbb{K}$, and $\mathbb{E}_\omega$ is the corresponding non-Archimedean Hilbert space to $\omega = (\omega_t)_{t\in\mathbb{N}}$.

*Example 19.* Suppose that the ground field $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$ and consider the linear operator $A$ on $\mathbb{E}_\omega$ defined by

$$Ae_s = \sum_{k=0}^{\infty} a_{k,s} e_k,$$

where

$$a_{ks} = \begin{cases} \dfrac{p^{s+k}}{1+p+p^2+\dots+p^s} & \text{if } k \le s \\ 0 & \text{if } k > s. \end{cases}$$

We also require that:

(1) $|\omega_t| \ge 1$ for each $t \in \mathbb{N}$,
(2) $\sup_{t\in\mathbb{N}} |\omega_t| \le M$ for some $M > 0$.

[1] An operator $A$ whose trace does exists is also called *nuclear*.

**Proposition 23.** *Under assumptions (1)-(2) above, the operator A defined above is Hilbert-Schmidt on* $\mathbb{E}_\omega$.

*Proof.* For each $s$, $|a_{ks}|\,|\omega_k|^{1/2} = p^{-(s+k)}\,|\omega_k|_p^{1/2} \leq M^{1/2} p^{-(s+k)}$, hence

$$\forall s,\ \lim_k |a_{ks}|\,|\omega_k|^{1/2} = 0.$$

Therefore, $A$ is well-defined.

Now

$$\|Ae_s\|^2 = \left(\max_{0\leq k\leq s} |\frac{p^{s+k}}{1+p+p^2+....+p^s}|\,.\,|\omega_k|^{\frac{1}{2}}\right)^2,$$

and hence

$$\|Ae_s\|^2 \leq M\left(\max_{0\leq k\leq s} |\frac{p^{s+k}}{1+p+p^2+....+p^s}|\right)^2.$$

Since $|\frac{p^{s+k}}{1+p+p^2+....+p^s}|_p = p^{-(k+s)}$ it follows that

$$\|Ae_s\|^2 \leq M\left[\max_{0\leq k\leq s} |\frac{p^{s+k}}{1+p+p^2+....+p^s}|_p\right]^2 = Mp^{-2s},$$

that is,

$$\frac{\|Ae_s\|^2}{|\omega_s|} \leq Mp^{-2s},$$

by (1).

Since the series $\sum_{s=0}^{+\infty} M\,p^{-2s}$ converges, hence $A \in B_2(\mathbb{E}_\omega)$.

*Example 20.* Suppose that the ground field $\mathbb{K} = \mathbb{Q}_p$ and that $\omega_s = p^{-s}$ for all $s \in \mathbb{N}$. For integers $m \geq 1$ and $n \geq 0$ let

$$A^{(m,n)} = \sum_{t,s\in\mathbb{N}} \frac{1}{\omega_t^m \omega_s^n}\left(e'_s \otimes e_t\right).$$

**Proposition 24.** *Under previous assumptions,* $A^{(m,n)}$ *is a Hilbert-Schmidt on* $\mathbb{E}_\omega$.

*Proof.* For all $s$, $\left\|A^{(m,n)}e_s\right\|^2 = \sup_{t\in\mathbb{N}} \frac{1}{|\omega_t|^{2m-1}|\omega_s|^{2n}}$. Now since $|\omega_t| = p^t \to \infty$ as $t \to \infty$, there exists a positive $M$ such that $\sup_i \frac{1}{|\omega_i|^{2m-1}} \leq M$ and it follows that

$$\frac{\left\|A^{(m,n)}e_s\right\|^2}{|\omega_s|} \leq \frac{M}{|\omega_s|^{2n+1}}.$$

Again, since $|\omega_s| = p^s \to \infty$ as $s \to \infty$ it follows that the series

$$\sum_s \frac{M}{|\omega_s|^{2n+1}} = \sum_s \frac{M}{p^{s(2n+1)}}$$

converges, hence $A^{(m,n)} \in B_2(\mathbb{E}_\omega)$.

*Example 21.* This example is a variation of the one produced in [24]. Let $A$ be the operator on $\mathbb{E}_\omega$ defined by

$$Ae_s = \lambda_s e_s + \sum_{t \neq s} \frac{e_t}{\omega_t}.$$

**Proposition 25.** *Suppose that* $|\omega_s| \geq 1$ *for all s, and consider a sequence* $(\lambda_s)_{s \in \mathbb{N}}$ *in* $(\mathbb{K}, |.|)$ *satisfying:* $|\lambda_s| \geq |\omega_s|^{-1/2}$*for all s, and* $\sum_s |\lambda_s|^2$ *converges. Then* $A \in B_2(\mathbb{E}_\omega)$.

*Proof.* Clearly, $A \in B(\mathbb{E}_\omega)$. Moreover, for $s$

$$\|Ae_s\|^2 = \max\left( |\lambda_s|^2 |\omega_s|, \sup_{j \neq s} \frac{1}{|\omega_j|} \right) = |\lambda_s|^2 |\omega_s|.$$

The result now follows from the fact that $\sum_s |\lambda_s|^2$ converges.

*Remark 15.* In the case of $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$, one can take $\omega_s = p^{-3s}$ and $\lambda_s = p^s$.

## 3.6 Open Problems

*Problem 1.* Interesting questions related to this chapter consists of developing a comprehensive spectral theory for self-adjoint bounded operators, normal operators, completely continuous operators, and Hilbert-Schmidt operators, as it had been done in the classical setting.

The following questions, raised in Diagana[20], still remain:

*Problem 2.* Let $T : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ be a bounded linear operator. Suppose that $T$ does not have an adjoint. Can we perturb $T$ so that the resulting operator has an adjoint?

*Problem 3.* Let $T$ be a bounded linear operator on $\mathbb{E}_\omega$. Is it possible to decompose $T$ as $T = A + B$, where $A \in B_0(\mathbb{E}_\omega)$ and $B$ does not have an adjoint?

*Problem 4.* Characterize the collection of all bounded linear operators on $\mathbb{E}_\omega$, which do not have adjoint.

*Problem 5.* Let $T$ be a Hilbert-Schmidt operator on $\mathbb{E}_\omega$. Define the square root $T^{1/2}$ of $T$? Does it exist an orthogonal basis $(\phi_t)_t$ for $\mathbb{E}_\omega$ such that

$$Tx = \sum_{t \in \mathbb{N}} \mu_t \frac{\langle x, \phi_t \rangle}{\omega_t} \,.\, \phi_t, \quad x \in \mathbb{E}_\omega,$$

where $\mu_t \in \mathbb{K}$, $\forall t \in \mathbb{N}$?

## 3.7 Bibliographical Notes

This chapter is entirely devoted to non-Archimedean bounded linear operators. Most of the results of this chapter can be found in Diagana et al. **[4, 12, 17]**, and Diarra **[24, 25]**.

*Chapter 4*

# Non-Archimedean Unbounded Linear Operators

## 4.1 Introduction

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of nonzero elements in a (complete) non-Archimedean field $(\mathbb{K}, |\cdot|)$ and let $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be the corresponding non-Archimedean Hilbert space (Section 3.2). This chapter is devoted to unbounded linear operators on free Banach spaces (respectively, on non-Archimedean Hilbert spaces $\mathbb{E}_\omega$). As for non-Archimedean bounded linear operators, some of the results go along with the classical line and others deviate from it. For the most part, the statements of the results are inspired by their classical counterparts. However their proofs may depend heavily on the non-Archimedean nature of both $\mathbb{E}_\omega$ and the ground field $\mathbb{K}$. As for non-Archimedean bounded operators on $\mathbb{E}_\omega$, we shall see that there exist unbounded linear operators, which do not have adjoint. Both the closedness and the self-adjointness of those unbounded linear operators will be screened. To deal with both the closedness and the self-adjoiness of those linear operators, we will equip the direct sum $\mathbb{E}_\omega \times \mathbb{E}_\omega$ with both a complete non-Archimedean norm and a (non-Archimedean) hilbertian structure. As in the classical context, we consider an unitary operator $\Gamma$ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, which yields a remarkable description the adjoint $A^*$ of $A$ in terms of $A$.

As illustrations, several examples will be discussed. In particular, special attention is paid to the so-called (unbounded) diagonal operator, which will play a crucial role to diagonalizing self-adjoint linear operators (see Chapter 6).

Throughout the rest of this chapter, $(\mathbb{K}, |\cdot|)$ denotes a complete non-Archimedean valued field.

## 4.2 Basic Definitions

Let $(\mathbb{E}, \|\cdot\|)$ and $(\mathbb{F}, |||\cdot|||)$ be free Banach spaces over $\mathbb{K}$ and let $(e_t)_{t\in\mathbb{N}}$, $(h_t)_{t\in\mathbb{N}}$ denote the canonical orthogonal bases associated with $\mathbb{E}$ and $\mathbb{F}$, respectively.

The next definition is crucial throughout the rest of the book and is due to Diagana **[17, 18]**.

**Definition 23.** An unbounded linear operator $A$ from $\mathbb{E}$ into $\mathbb{F}$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathbb{E}$ (called the domain of $A$) and a (possibly not continuous) linear transformation $A : D(A) \mapsto \mathbb{F}$. The domain $D(A)$ contains the base $(e_i)_{i\in\mathbb{N}}$ and consists of all $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}$ such that $Au = \sum_{i\in\mathbb{N}} u_i A e_i$ converges in $\mathbb{F}$, that is

$$\begin{cases} D(A) := \{u = (u_t)_{t\in\mathbb{N}} \in \mathbb{E} : \ \lim_{t\to\infty} |u_t| \,.\, |||Ae_t||| = 0\}, \\ \\ A = \sum_{t,s\in\mathbb{N}} a_{ts}\, e'_s \otimes h_t, \quad \forall s \in \mathbb{N}, \ \lim_{t\to\infty} |a_{t,s}| \,.\, |||h_t||| = 0. \end{cases}$$

The collection of those unbounded linear operators is denoted by $U(\mathbb{E}, \mathbb{F})$ and $U(\mathbb{E})$ in the case when $\mathbb{E} = \mathbb{F}$.

### 4.2.1 Example

*Example 22.* Let $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$ and suppose that $\mathbb{E} = \mathbb{F} = C(\mathbb{Z}_p, \mathbb{Q}_p)$, is the non-Archimedean Banach space appearing in Examples 9 and 13 equipped with its corresponding sup norm.

Let $\gamma = (\gamma_t)_{t\in\mathbb{N}} : \mathbb{Z}_p \mapsto \mathbb{Q}_p$ be an arbitrary sequence of functions, not necessarily continuous. Define

$$\begin{cases} D(M_\gamma) := \{u(x) = \sum_{t=0}^{\infty} u_t f_t(x) \in \mathbb{E} : \lim_{t\to\infty} |u_t| \,.\, \|M_\gamma f_t\|_\infty = 0\}, \\ \\ M_\gamma u(x) := \sum_{t=0}^{\infty} u_t \gamma_t(x) f_t(x), \ \ \forall \ u(x) = \sum_{t=0}^{\infty} u_t f_t(x) \in D(M_\gamma). \end{cases}$$

It can be easily checked that the linear operator $M_\gamma$ is well-defined. Moreover, depending on the boundedness or not of the expression $\sup_{t\in\mathbb{N}} \|\gamma_t\|_\infty$, the operator $M_\gamma$ may or may not be bounded[1].

Throughout the rest of this chapter we take $\mathbb{E} = \mathbb{F} = \mathbb{E}_\omega$ with $\omega = (\omega_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ being a sequence of nonzero.

We have previously seen that if $A \in B(\mathbb{E}_\omega)$ then $D(A) = \mathbb{E}_\omega$. Here, we will see that the domains of most of operators that we consider differ from $\mathbb{E}_\omega$. As for bounded operators, there are elements of $U(\mathbb{E}_\omega)$, which do not have adjoint. In the next definition, we state some necessary and sufficient conditions, which do guarantee the existence of the adjoint.

[1] If $\sup_{t\in\mathbb{N}} \|\gamma_t\|_\infty = \infty$, then $M_\gamma$ is an unbounded linear operator on $C(\mathbb{Z}_p, \mathbb{Q}_p)$. Otherwise, it is a bounded linear operator.

### 4.2.2 Existence of the Adjoint

**Definition 24.** A linear operator $A = \sum_{t,s} a_{ts}\ e'_s \otimes e_t$ in $U(\mathbb{E}_\omega)$ is said to have an adjoint $A^* \in U(\mathbb{E}_\omega)$ if and only if:

$$\lim_{s\to\infty} |\omega_s|^{-1/2} . |a_{ts}| = 0, \ \ \forall t \in \mathbb{N}. \tag{4.2.1}$$

Furthermore, under assumption (4.2.1), the adjoint $A^*$ of $A$ is uniquely expressed by

$$\begin{cases} D(A^*) := \{v = (v_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{t\to\infty} |v_t|\ \|A^* e_t\| = 0\}, \\ \\ A^* = \sum\limits_{t,s\in\mathbb{N}} a^*_{ts}\ e'_s \otimes e_t, \quad \forall s \in \mathbb{N},\ \lim\limits_{t\to\infty} |a^*_{ts}|\ |\omega_t|^{\frac{1}{2}} = 0, \end{cases}$$

where $a^*_{ts} = \omega_t^{-1}\omega_s a_{st}$.

Set

$$U_0(\mathbb{E}_\omega) = \left\{ A = \sum_{t,s} a_{ts} e'_s \otimes e_t \in U(\mathbb{E}_\omega) : \lim_{s\to\infty} \frac{|a_{ts}|}{|\omega_s|^{1/2}} = 0,\ \ \forall t \in \mathbb{N} \right\}.$$

Clearly, $B_0(\mathbb{E}_\omega) \subset U_0(\mathbb{E}_\omega)$.

*Remark 16.* In the classical context, if $B$ is an unbounded linear operator on a Hilbert space $H$, then $(B^*)^* = \overline{B}$ ($\overline{B}$ being the closure of $B$). In the non-Archimedean setting, it is not difficult to check that if $A = \sum_{t,s\in\mathbb{N}} a_{ts} e'_s \otimes e_t \in U_0(\mathbb{E}_\omega)$, then $A^* \in U_0(\mathbb{E}_\omega)$. Moreover, $(a^*_{ts})^* = a_{ts}$, $\forall t, s \in \mathbb{N}$, and hence

$$(A^*)^* = A.$$

### 4.2.3 Examples of Unbounded Operators With no Adjoint

*Example 23.* Set $\mathbb{K} = \mathbb{Q}_p$, the field of $p$-adic numbers endowed with the $p$-adic absolute value $|\cdot|$ and let $\omega_t = p^{3t}$ so that $|\omega_t|^{\frac{1}{2}} = p^{-\frac{3}{2}t}$. Define $A = \sum_{t,s} a_{ts} . (e'_s \otimes e_t)$ by its coefficients:

$$a_{ts} = \begin{cases} p^{-s} & \text{if} \ \ t < s \\ 1 & \text{if} \ \ t = s \\ p^{-t} & \text{if} \ \ t > s. \end{cases}$$

**Proposition 26.** *The linear operator $A$ defined above is in $U(\mathbb{E}_\omega)$ and does not have an adjoint.*

*Proof.* Since $\forall s$, $\lim\limits_{t} |a_{ts}||\omega_t|^{\frac{1}{2}} = \lim\limits_{t>s} p^t p^{-\frac{3}{2}t} = 0$ it is then clear that $A = \sum_{ts} a_{ts} . (e'_s \otimes e_t)$ is well-defined. Furthermore, $\forall t, s \in \mathbb{N}$,

$$\lambda_{ts} := \frac{|a_{ts}||\omega_t|^{\frac{1}{2}}}{|\omega_s|^{\frac{1}{2}}} = \begin{cases} p^{\frac{3}{2}(s-t)+s} & \text{if } t<s \\ 1 & \text{if } t=s \\ p^{t+\frac{3}{2}s-\frac{3}{2}t} & \text{if } t>s. \end{cases}$$

Consequently,

$$\forall t,s,\ \ \lambda_{t,s} := \frac{|a_{ts}||\omega_t|^{1/2}}{|\omega_s|^{1/2}} \geq \begin{cases} p^{s} & \text{if } t<s \\ 1 & \text{if } t=s \\ p^{-\frac{3}{2}t} & \text{if } t>s. \end{cases}$$

Clearly, $\|A\| := \sup_{t,s} \lambda_{t,s} = \infty$, hence $A \in U(\mathbb{E}_\omega)$.

To complete the proof we have to show that $\forall t,\ \lim_s \frac{|a_{ts}|}{|\omega_s|^{\frac{1}{2}}} \neq 0$. Indeed,

$$\forall t,\ \lim_s \frac{|a_{ts}|}{|\omega_s|^{\frac{1}{2}}} = \lim_{s>t} p^s p^{\frac{3}{2}s} = \infty,$$

hence the adjoint of $A$ does not exist.

Similarly, taking $b_{ts} = p^{-t-s}$, for all $t,s \in \mathbb{N}$, and $\omega_t = p^{3t}$ for each $t \in \mathbb{N}$, one can easily check that:

$$B = \sum_{t,s\in\mathbb{N}} p^{-t-s} e'_s \otimes e_t \notin U_0(\mathbb{E}_\omega).$$

## 4.3 Closed Linear Operators on $\mathbb{E}_\omega$

Let $A \in U(\mathbb{E}_\omega)$. As in the classical setting, we define the graph of the linear operator $A$ by

$$\mathcal{G}(A) := \{(x, Ax) \in \mathbb{E}_\omega \times \mathbb{E}_\omega :\ x \in D(A)\}.$$

**Definition 25.** An operator $A \in U(\mathbb{E}_\omega)$ is said to be closed if its graph is a closed subspace in $\mathbb{E}_\omega \times \mathbb{E}_\omega$. The operator $A$ is said to be closable if it has a closed extension.

As in the classical theory of unbounded linear operators we characterize the closedness of an operator $A \in U(\mathbb{E}_\omega)$ as follows: $\forall u_t \in D(A)$ such that $\|u - u_t\| \mapsto 0$ and $\|Au_t - v\| \mapsto 0$ $(v \in \mathbb{E}_\omega)$ as $t \mapsto \infty$, then $u \in D(A)$ and $Au = v$.

*Remark 17.* Note that if $A \in B(\mathbb{E}_\omega)$, then it is closed. Indeed since $A$ is bounded, $D(A) = \mathbb{E}_\omega$. Moreover if $u_t \in \mathbb{E}_\omega$ such that $u_t \mapsto u$ on $\mathbb{E}_\omega$ as $t \mapsto \infty$, then by the boundedness of $A$ it follows that $Au_t \mapsto Au$ as $t \mapsto \infty$, that is, $(u_t, Au_t) \mapsto (u, Au)$ as $t \mapsto \infty$ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, hence $\mathcal{G}(A)$ is closed.

Let $C(\mathbb{E}_\omega)$ denote the collection of closed linear operators $A \in U(\mathbb{E}_\omega)$ . Obviously, $B(\mathbb{E}_\omega) \subset C(\mathbb{E}_\omega)$.

*Remark 18.* As in the classical setting, it is not difficult to see that if $A \in C(\mathbb{E}_\omega)$ and if $B \in B(\mathbb{E}_\omega)$, then $A+B \in C(\mathbb{E}_\omega)$.

**Definition 26.** If $A, B$ are (possibly unbounded) linear operators on $\mathbb{E}_\omega$ such that $D(A) \subset D(B)$ and $Ax = Bx$ for each $x \in D(A)$, then $B$ is said to be an extension of $A$ and write $A \subset B$.

Let $A$ and $B$ be (possibly unbounded) linear operators on $\mathbb{E}_\omega$; their algebraic sum and product are respectively defined by

$$\begin{cases} D(A+B) = D(A) \cap D(B) \\ \\ (A+B)u = Au + Bu, \quad \forall u \in D(A) \cap D(B), \end{cases}$$

and

$$\begin{cases} D(AB) = \{u \in D(B) : Bu \in D(A)\} \\ \\ (AB)u = A(Bu), \quad \forall u \in D(AB). \end{cases}$$

The domain $D(A) \cap D(B)$ of the algebraic sum $A+B$ has to be watched with care. Indeed, it may happen $D(A) \cap D(B) = \{0\}$. In this event, $A+B$ is trivial.

Let $A : D(A) \mapsto \mathbb{E}_\omega$ be an unbounded linear operator on $\mathbb{E}_\omega$. Note that in addition to the topology of $\mathbb{E}_\omega$, $D(A)$ can be provided with another norm that we will call non-Archimedean *graph norm* defined by:

$$\|u\|_{D(A)} = \max(\|u\|, \|Au\|), \quad \forall u \in D(A).$$

It is clear that the non-Archimedean graph norm is stronger than the norm of $\mathbb{E}_\omega$. Moreover, those two norms are equivalent if and only if $A \in B(\mathbb{E}_\omega)$.

**Proposition 27.** *Let $A : D(A) \mapsto \mathbb{E}_\omega$ be an unbounded linear operator on $\mathbb{E}_\omega$. Then $(D(A), \|\cdot\|_{D(A)})$ is a non-Archimedean Banach space if and only if $A$ is closed.*

*Proof.* It is easy to see that $(D(A), \|\cdot\|_{D(A)})$ is a non-Archimedean normed vector space. Suppose that $A$ is closed. Let $(u_t)_{t\in\mathbb{N}} \subset D(A)$ be a Cauchy sequence. Therefore, for each $\varepsilon > 0$ there exists $T_0$ such that:

$$\|u_t - u_s\|_{D(A)} = \max(\|u_t - u_s\|, \|Au_t - Au_s\|) \leq \varepsilon$$

whenever $t \geq s \geq T_0$.

Consequently, both $(u_t)_{t\in\mathbb{N}}$ $(Au_t)_{t\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{E}_\omega$. Since $\mathbb{E}_\omega$ is complete, $u_t \to u$, $Au_t \to \xi$ as $t \to \infty$. Clearly, $u \in D(A)$ and $Au = \xi$ because of the closedness of $A$, and hence $\|u_t - u\|_{D(A)} \mapsto 0$ as $t \to \infty$, that is, $(D(A), \|\cdot\|_{D(A)})$ is complete.

Now suppose that $(D(A), \|\cdot\|_{D(A)})$ is complete and let $(u_t)_{t\in\mathbb{N}} \subset D(A)$ such that $\|u_t - u\|$ and $\|Au_t - \xi\| \to 0$ as $t \to \infty$. We want to show that $u \in D(A)$ and $Au = \xi$. Clearly,

$$\|u_t - u_s\|_{D(A)} = \max(\|u_t - u_s\|, \|Au_t - Au_s\|) \to 0 \ \text{ as } \ t,s \to \infty,$$

and hence $(u_t)_{t\in\mathbb{N}}$ is a Cauchy sequence in $(D(A), \|\cdot\|_{D(A)})$. In view of the above, there exists $v \in D(A)$ such that

$$\|u_t - v\|_{D(A)} = \max(\|u_t - v\|, \|Au_t - Av\|) \to 0 \ \text{ as } \ t \to \infty.$$

In particular, $\|u_t - v\|, \|Au_t - Av\| \to 0$ as $t \to \infty$, and therefore $u = v \in D(A)$ and $Av = Au = \xi$, because of the uniqueness of the limit.

**Definition 27.** An operator $A \in U(\mathbb{E}_\omega)$ is said to be symmetric if $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in D(A)$.

**Definition 28.** An operator $A \in U_0(\mathbb{E}_\omega)$ is said to be self-adjoint if $D(A) = D(A^*)$ and $Au = A^*u$ for each $u \in D(A)$.

Clearly, every self-adjoint operator is symmetric.

We shall prove the following important theorem:

**Theorem 12.** *Let $A \in U_0(\mathbb{E}_\omega)$. Then its adjoint $A^*$ is a closed linear operator. In particular, if $A$ is self-adjoint, then it is closed.*

Let $\Gamma$ be the bounded linear operator which goes from $\mathbb{E}_\omega \times \mathbb{E}_\omega$ into itself defined by:

$$\Gamma(u,v) := (-v,u), \ \forall (u,v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega.$$

Clearly, $\Gamma$ satisfies:

(i) $\Gamma^2(u,v) = -(u,v), \ \ \forall (u,v) \in \mathbb{E}_\omega \times \mathbb{E}_\omega$;

(ii) $\Gamma^2(M) = M$ for any subspace $M$ of $\mathbb{E}_\omega \times \mathbb{E}_\omega$.

From (i) it follows that: $\Gamma^2 = -I$ ($\Gamma$ is an unitary operator), where $I$ is the identity operator of $\mathbb{E}_\omega \times \mathbb{E}_\omega$.

**Theorem 13.** *If $A \in U_0(\mathbb{E}_\omega)$, then $\mathcal{G}(A^*) = [\Gamma\mathcal{G}(A)]^\perp$, where $[\Gamma\mathcal{G}(A)]^\perp$ is the orthogonal complement of $[\Gamma\mathcal{G}(A)]$.*

*Proof.* (Theorem 13). Since the adjoint $A^*$ of $A$ does exist, one defines its graph by:

$$\mathcal{G}(A^*) := \{(x, A^*x) \in \mathbb{E}_\omega \times \mathbb{E}_\omega : \ x \in D(A^*)\}.$$

Thus for $(x,y) \in D(A^*) \times D(A)$ one has

$$\begin{aligned}
\langle (x,A^*x), \Gamma(y,Ay)\rangle_2 &= \langle (x,A^*x), (-Ay,y)\rangle_2 \\
&= -\langle x, Ay\rangle + \langle A^*x, y\rangle \\
&= 0,
\end{aligned}$$

hence $\mathcal{G}(A^*) \subset [\Gamma\mathcal{G}(A)]^{\perp}$.

Conversely, if $(x,y) \in [\Gamma\mathcal{G}(A)]^{\perp}$, $\forall z \in D(A)$, then,

$$\begin{aligned} 0 &= \langle (x,y),(-Az,z)\rangle_2 \\ &= -\langle x,Az\rangle + \langle y,z\rangle. \end{aligned}$$

It follows that $\langle Az,x\rangle = \langle z,y\rangle$. Now by uniqueness of the adjoint, we obtain that: $x \in D(A^*)$ and $A^*x = y$, hence $(x,y) \in \mathcal{G}(A^*)$.

*Proof.* (Theorem 12). Since the adjoint of $A^*$ of $A$ does exist and that $\Gamma\mathcal{G}(A)$ is a subspace of $\mathbb{E}_\omega \times \mathbb{E}_\omega$, then using Theorem 13 it follows that $\mathcal{G}(A^*) = [\Gamma\mathcal{G}(A)]^{\perp}$ is closed, hence $A^*$ is closed.

## 4.4 Diagonal Operators on $\mathbb{E}_\omega$

Let $\omega = (\omega_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of nonzero elements and let $\mathbb{E}_\omega$ be its corresponding non-Archimedean Hilbert space. Let $(\lambda_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of elements such that

$$\lim_{t\to\infty} |\lambda_t| = \infty. \tag{4.4.1}$$

Define the diagonal operator $A \in U(\mathbb{E}_\omega)$ by

$$D(A) = \{x = (x_t) \subset \mathbb{K} : \lim_t |\lambda_t| \,.\, |x_t| \,.\, \|e_t\| = 0\},$$

and

$$Ax = \sum_{t\in\mathbb{N}} \lambda_t x_t e_t, \quad \text{for each } \ x = \sum_{t\in\mathbb{N}} x_t e_t \in D(A).$$

**Proposition 28.** *Under assumption (4.4.1), the diagonal operator A defined above is a (unbounded) self-adjoint operator on* $\mathbb{E}_\omega$*. Furthermore,* $\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_t, \ \forall t \in \mathbb{N}\}$*, and*

$$\|(A-\lambda)^{-1}\| = \sup_{t\in\mathbb{N}} \left(\frac{1}{|\lambda_t - \lambda|}\right)$$

*for each* $\lambda \in \rho(A)$.

*Proof.* First of all, let us make sure that the operator $A$ is well-defined. For that, note that $|a_{tt}| = |\lambda_t|$ and that $|a_{ts}| = 0$ if $t \neq s$, and

$$\lim_t \ |a_{ts}| \ |\omega_t|^{1/2} = \lim_{t>s} \ |a_{ts}| \ |\omega_t|^{1/2} = 0,$$

and hence $A$ is well-defined.

Now $\dfrac{|a_{ts}| \ |\omega_t|^{1/2}}{|\omega_s|} = |\lambda_t|$ if $t = s$ and 0 if $t \neq s$. It follows that

$$\|A\| := \sup_{t,s} \frac{|a_{ts}| \ |\omega_t|^{1/2}}{|\omega_s|} = \sup_t |\lambda_t| = \infty,$$

and hence $A \in U(\mathbb{E}_\omega)$.

Let us show that the adjoint $A^*$ of $A$ does exist. This is actually obvious since

$$\forall t \in \mathbb{N},\ \lim_{s} |a_{ts}|\ |\omega_s|^{-1/2} = \lim_{s>t} |a_{ts}|\ |\omega_s|^{-1/2} = 0.$$

Clearly, the adjoint $A^*$ is defined by $A^* = \sum_{ts} b_{ts}\ e'_s \otimes e_t$, where $b_{ts} = w_t^{-1} w_s a_{st} = a_{ts}$ for all $t, s \in \mathbb{N}$, and hence $A = A^*$.

To complete the proof we have to compute elements of the resolvent $\rho(A)$ of $A$. We want to find all $\lambda \in \mathbb{K}$ such that

$$(A - \lambda I)x = y, \tag{4.4.2}$$

where $x = \sum_t x_t e_t \in D(A) = D(A - \lambda I)$ and $y = \sum_t y_t e_t \in \mathbb{E}_\omega$.

Considering (4.4.2) on $(e_t)_{t \in \mathbb{N}}$ and using the fact $A$ is self-adjoint it follows that

$$\forall t \in \mathbb{N},\ (\lambda_t - \lambda)\,.\,\langle e_t, x\rangle = \langle e_t, y\rangle.$$

Equivalently, $\forall t \in \mathbb{N}$,

$$(\lambda_t - \lambda)\,.\,\omega_t x_t = \omega_t y_t. \tag{4.4.3}$$

For all $\lambda_t \neq \lambda$, (4.4.2) has a unique solution $x$. Moreover,

$$x = (A - \lambda)^{-1} y = \sum_{t \in \mathbb{N}} \frac{y_t}{\lambda_t - \lambda} e_t. \tag{4.4.4}$$

Let us show that $x = (A - \lambda)^{-1} y$ given above is well-defined. For that it is sufficient to prove that

$$\lim_{t \to \infty} \frac{|y_t|}{|\lambda_t - \lambda|}\ \|e_t\| = 0.$$

Using (4.4.1) it easily follows that the sequence $\left(\dfrac{1}{|\lambda_t - \lambda|}\right)_{t \in \mathbb{N}}$ is bounded, and hence

$$\lim_{t \to \infty} \frac{|y_t|}{|\lambda_t - \lambda|}\ \|e_t\| = 0.$$

It remains to find conditions on $\lambda$ so that $x$ defined above belongs to $D(A)$. For that, it is sufficient to show that

$$\lim_{t \to \infty} \frac{|y_t|}{|\lambda_t - \lambda|} \|Ae_t\| = \lim_{t \to \infty} \frac{|\lambda_t|}{|\lambda_t - \lambda|}\ |y_t|\ \|e_t\| = 0.$$

Indeed, since $\lim_{t \to \infty} |y_t|\ \|e_t\| = 0$,

$$\begin{aligned} 0 &\leq \lim_{t \to \infty} \frac{|y_t|}{|\lambda_t - \lambda|} \|Ae_t\| \\ &\leq \lim_{t \to \infty} \frac{|\lambda_t|}{|\ |\lambda_t| - |\lambda|\ |}\ .\ \lim_{t} |y_t|\ \|e_t\| \\ &= 0. \end{aligned}$$

From (4.4.4) it follows that for each $t \in \mathbb{N}$, $\|(A-\lambda)^{-1}e_t\| = \dfrac{\|e_s\|}{|\lambda_t - \lambda|}$, in other words,

$$\|(A-\lambda)^{-1}\| = \sup_{t\in\mathbb{N}} \left(\frac{1}{|\lambda_t - \lambda|}\right),$$

and hence $(A-\lambda)^{-1} \in B(\mathbb{E}_\omega)$.

In summary, the resolvent $\rho(A)$ of $A$ given by:

$$\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_t, \ \forall t \in \mathbb{N}\}.$$

*Remark 19.* Note that $D(A)$, the domain of the diagonal operator $A$ may or may not be equal the whole $\mathbb{E}_\omega$. Indeed, take, for each $t \in \mathbb{N}$, $\mu_t \in \mathbb{K} - \{0\}$, $\omega_t = \mu_t^2$, and $\tilde{x} = (\tilde{x}_t)_{t\in\mathbb{N}}$ where $\tilde{x}_t = \dfrac{1}{\lambda_t \ \mu_t}$ for all $t \in \mathbb{N}$. Clearly, $\tilde{x} \in \mathbb{E}_\omega$ since

$$\lim_{t\to\infty} |\tilde{x}_t| \|e_t\| = \lim_{t\to\infty} \frac{1}{|\lambda_t|} = 0,$$

by (4.4.1). Meanwhile, one can easily see that $\tilde{x} \notin D(A)$ since

$$\lim_{t\to\infty} \ |\tilde{x}_t| \,.\, |\lambda_t| \,.\, \|e_t\| = 1 \neq 0.$$

## 4.5 Open Problems

*Problem 1.* Interesting questions related to this chapter consists of developing a comprehensive spectral theory for self-adjoint and normal (unbounded) linear operators.

*Problem 2.* Let $T : D(T) \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ be a unbounded linear operator. Suppose that $T$ does not have an adjoint. How can we perturb $T$ so that the resulting operator has an adjoint?

*Problem 3.* Let $T$ be an unbounded linear operator on $\mathbb{E}_\omega$. Is it possible to decompose $T$ as: $T = A + B$, where $A \in U_0(\mathbb{E}_\omega)$ and $B$ does not have an adjoint?

*Problem 4.* Characterize (spectrally) the collection of all unbounded linear operators on $\mathbb{E}_\omega$, which do not have adjoint?

*Problem 5.* Let $A, B$ be unbounded linear operators on $\mathbb{E}_\omega$. Find necessary and sufficient conditions so that $A + B \in C(\mathbb{E}_\omega)$.

## 4.6 Bibliographical Notes

This chapter is entirely devoted to non-Archimedean unbounded linear operator. Most of results (published or not) of this chapter are due to Diagana**[17, 18]**.

*Chapter 5*

# Non-Archimedean Bilinear Forms

## 5.1 Introduction

In the classical setting, Bounded and unbounded sesquilinear forms on Hilbert spaces play an crucial role in several fields such as quantum mechanics, mathematical physics, variational analysis, symplectic geometry, see, e.g., de Bivar-Weinholtz and Lapidus **[5]**, Diagana **[19]**, Johnson and Lapidus **[34]**, and Kato**[35]**.

The present chapter examines a preliminary work of the author on non-Archimedean counterparts of the classical bilinear forms on Hilbert spaces. To do so, we first introduce and study bounded and unbounded (symmetric) bilinear forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$; we then call those forms non-Archimedean bilinear forms. Secondly, we emphasis on the closedness of these bilinear forms in connection with the associated non-Archimedean quadratic forms as it had been done in the classical setting in Kato **[35, Chapter VI, p. 313]**. In particular, some sufficient conditions for the closedness of the form sum of the so-called quadratically disjoint closed bilinear forms, are given. In addition to the above, we also discuss the construction of non-Archimedean Hilbert spaces through bilinear forms.

At the end of this chapter we discuss on a new representation theorem, which basically says that under some suitable assumptions, each non-degenerate (unbounded) bilinear form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ is representable by a linear operator $A$ under some. Furthermore, when such a bilinear form is symmetric, then the operator $A$ is self-adjoint.

If $\omega = (\omega_t)_{t\in\mathbb{N}}$ is a sequence of nonzero elements in $\mathbb{K}$, then we let $\mathbb{E}_\omega$ denote its corresponding non-Archimedean Hilbert space and $(e_t)_{t\in\mathbb{N}}$ denotes the canonical orthogonal base associated with $\mathbb{E}_\omega$.

## 5.2 Basic Definitions

### 5.2.1 Continuous Linear Functionals on $\mathbb{E}_\omega$

**Definition 29.** *A linear functional* $\varphi : \mathbb{E}_\omega \mapsto \mathbb{K}$ *is said to be continuous if there exists* $K \geq 0$ *such that*

$$|\varphi(u)| \leq K \,.\, \|u\|, \;\; u \in \mathbb{E}_\omega. \tag{5.2.1}$$

*The smallest K such that (5.2.1) holds is called the norm of the continuous linear functional* $\varphi$ *and is defined by*

$$|||\varphi||| = \sup_{u \neq 0} \left( \frac{|\varphi(u)|}{\|u\|} \right).$$

The space of all continuous linear functionals on $\mathbb{E}_\omega$ is denoted $\mathbb{E}_\omega^*$ and called the (topological) dual of $\mathbb{E}_\omega$. The space $(\mathbb{E}_\omega^*, ||| \cdot |||)$ is a Banach space over $\mathbb{K}$.

**Proposition 29.** *Let* $\varphi : \mathbb{E}_\omega \mapsto \mathbb{K}$ *be a continuous linear functional. Then its norm* $|||\varphi|||$ *can be explicitly expressed as*

$$|||\phi||| = \sup_{i \in \mathbb{N}} \left( \frac{|\varphi(e_i)|}{\|e_i\|} \right).$$

*Proof.* Obviously, $|||\phi||| \geq \sup_{i \in \mathbb{N}} \left( \frac{|\varphi(e_i)|}{\|e_i\|} \right)$, by the definition of the norm $|||\varphi|||$.

Now let $u \neq 0$,

$$\begin{aligned} |\varphi(u)| &= \left| \sum_{i=0}^{\infty} \varphi(e_i)\, u_i \right| \\ &\leq \sup_{i \in \mathbb{N}} \left( |\varphi(e_i)| \,.\, |u_i| \right) \\ &= \sup_{i \in \mathbb{N}} \left( \frac{|\varphi(e_i)|(|u_i| \,.\, \|e_i\|)}{\|e_i\|} \right) \\ &\leq \|u\| \,.\, \sup_{i \in \mathbb{N}} \left( \frac{|\varphi(e_i)|}{\|e_i\|} \right), \end{aligned}$$

and hence

$$|||\varphi||| \leq \sup_{i \in \mathbb{N}} \left( \frac{|\varphi(e_i)|}{\|e_i\|} \right).$$

One completes the proof by combining the first and the latest inequalities.

The next theorem is a non-archimedean version of the well-known Riesz representation theorem [35].

**Theorem 14.** *Let* $\varphi : \mathbb{E}_\omega \mapsto \mathbb{K}$ *be a linear functional such that*

$$\lim_{i \to \infty} \frac{\|\varphi(e_i)\|}{\|e_i\|} = 0. \tag{5.2.2}$$

*Then there exists a unique* $u_0 \in \mathbb{E}_\omega$ *such that*

$$\varphi(u) = \langle u, u_0 \rangle, \;\; \textit{for all } u \in \mathbb{E}_\omega.$$

*Moreover,* $|||\varphi||| = \|u_0\|$.

*Proof.* Obviously, (5.2.2) yields $\varphi$ is continuous as $|||\varphi||| = \sup_{i\in\mathbb{N}} \frac{\|\varphi(e_i)\|}{\|e_i\|} < \infty$.

Let $u = \sum_{i\in\mathbb{N}} u_i e_i \in \mathbb{E}_\omega$. Now, $\phi(u) = \sum_{i\in\mathbb{N}} u_i \varphi(e_i)$ is well-defined. Indeed, since $u \in \mathbb{E}_\omega$, $\lim_{i\to\infty} |u_i|\|e_i\| = 0$, and hence

$$\lim_{i\to\infty} |u_i\varphi(e_i)| \leq \|u\| \,.\, \lim_{i\to\infty} \frac{\|\varphi(e_i)\|}{\|e_i\|} = 0,$$

by (5.2.2).

Now, set $u_0 = \sum_{i\in\mathbb{N}} \frac{\phi(e_i)}{\omega_i} e_i$. Using (5.2.2), one can easily see that $u_0 \in \mathbb{E}_\omega$. Moreover, $\varphi(u) = \langle u, u_0 \rangle$ for each $u \in \mathbb{E}_\omega$.

Suppose that there exists another $v_0 \in \mathbb{E}_\omega$ such that $\varphi(u) = \langle u, v_0 \rangle$ for each $u \in \mathbb{E}_\omega$. Then, $\langle u_0 - v_0, u \rangle = 0$ for each $u \in \mathbb{E}_\omega$, that is, $u_0 - v_0 \perp \mathbb{E}_\omega$. In particular, $\langle u_0 - v_0, e_i \rangle = 0$ for each $i \in \mathbb{N}$, that is, all coordinates of $u_0 - v_0$ in the canonical base $(e_i)_{i\in\mathbb{N}}$ of $\mathbb{E}_\omega$ are zero, and hence $u_0 = v_0$.

Now

$$\|u_0\| := \sup_{i\in\mathbb{N}} \left\| \frac{\phi(e_i)}{\omega_i} e_i \right\| = \sup_{i\in\mathbb{N}} \frac{\|\varphi(e_i)\|}{\|e_i\|} = |||\varphi|||.$$

**Definition 30.** A mapping $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is called a non-Archimedean bilinear form if $u \mapsto \phi(u,v)$ is linear for each $v \in \mathbb{E}_\omega$ and $v \mapsto \phi(u,v)$ linear for each $u \in \mathbb{E}_\omega$.

One can easily see that if $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is a non-Archimedean bilinear form over $\mathbb{E}_\omega \times \mathbb{E}_\omega$, then, for all $u = (u_t)_{t\in\mathbb{N}}, v = (v_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega$,

$$\phi(u,v) = \sum_{t,s=0}^{\infty} \sigma_{ts} u_t \,.\, v_s, \quad \forall s \in \mathbb{N}, \ \lim_{t\to\infty} |u_t||\sigma_{ts}|^{1/2} = 0, \tag{5.2.3}$$

where $\sigma_{ts} = \phi(e_t, e_s)$ for all $t, s \in \mathbb{N}$.

### 5.2.2 Bounded Bilinear Forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$

**Definition 31.** A non-Archimedean bilinear form $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be bounded if there exists $M \geq 0$ such that

$$|\phi(u,v)| \leq M \,.\, \|u\| \,.\, \|v\| \tag{5.2.4}$$

for all $u, v \in \mathbb{E}_\omega$.

*The smallest M such that (3.3.1) holds is called the norm of the bilinear form* $\phi$ *and is defined by*

$$\|\phi\| = \sup_{u,v\neq 0} \left( \frac{|\phi(u,v)|}{\|u\| \,.\, \|v\|} \right).$$

*Example 24.* Let $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ be a bounded linear operator. Setting

$$\Phi(u,v) = \langle Au, v\rangle \text{ for all } u,v \in \mathbb{E}_\omega,$$

where $\langle \cdot,\cdot\rangle$ is the inner product given in (2.3.2), it is then clear that $\Phi$ is a bounded non-Archimedean bilinear form over $\mathbb{E}_\omega \times \mathbb{E}_\omega$.

**Proposition 30.** *Let $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ be a bounded bilinear form. Then its norm $\|\phi\|$ can be explicitly expressed as*

$$\|\phi\| = \sup_{i,j\in\mathbb{N}} \left( \frac{|\phi(e_i,e_j)|}{\|e_i\| \,.\, \|e_j\|} \right).$$

*Proof.* The inequality, $\|\phi\| \geq \sup_{i,j\in\mathbb{N}} \left( \frac{|\phi(e_i,e_j)|}{\|e_i\| \,.\, \|e_j\|} \right)$, is a straightforward consequence of the definition of the norm $\|\phi\|$ of $\phi$.

Now suppose $u,v \neq 0$. In view of the above, one has

$$\begin{aligned}
|\phi(u,v)| &= \left| \sum_{i,j=0}^{\infty} \phi(e_i,e_j)\, u_i v_j \right| \\
&\leq \sup_{i,j\in\mathbb{N}} \left( |\phi(e_i,e_j)| \,.\, |u_i| \,.\, |v_j| \right) \\
&= \sup_{i,j\in\mathbb{N}} \left( \frac{|\phi(e_i,e_j)| (|u_i| \,.\, \|e_i\|)\, (|v_j| \,.\, \|e_j\|)}{\|e_i\| \,.\, \|e_j\|} \right) \\
&\leq \|u\| \,.\, \|v\| \,.\, \sup_{i,j\in\mathbb{N}} \left( \frac{|\phi(e_i,e_j)|}{\|e_i\| \,.\, \|e_j\|} \right),
\end{aligned}$$

and hence

$$\|\phi\| \leq \sup_{i,j\in\mathbb{N}} \left( \frac{|\phi(e_i,e_j)|}{\|e_i\| \,.\, \|e_j\|} \right).$$

One completes the proof by combining the first and the latest inequalities.

### 5.2.3 Unbounded Bilinear Forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$

We now introduce non-Archimedean analogues of the so-called *unbounded* bilinear forms.

**Definition 32.** A mapping $\Psi : D(\Psi) \times D(\Psi) \subset \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is called a non-Archimedean (unbounded) bilinear form if $u \mapsto \Psi(u,v)$ is linear for each $v \in D(\Psi)$ and $v \mapsto \Psi(u,v)$ linear for each $u \in D(\Psi)$, where

$$\begin{cases}
D(\Psi) := \{u = (u_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \lim\limits_{t\to\infty} |u_t| \,.\, |\Psi(e_t,e_t)|^{1/2} = 0\}, \\[2ex]
\Psi(u,v) = \sum\limits_{t,s=0}^{\infty} \sigma_{ts}\, u_t v_s, \quad \forall s \in \mathbb{N}, \ \lim\limits_{t\to\infty} |u_t| \,.\, |\sigma_{ts}|^{1/2} = 0
\end{cases}$$

for all $u,v \in D(\Psi)$, where $\sigma_{ts} = \Psi(e_t,e_s)$ for all $t,s \in \mathbb{N}$.

The subspace $D(\Psi)$ will be called the *domain* of the non-Archimedean bilinear form $\psi$. One says that $\Psi$ is densely defined if $D(\Psi)$ is dense in $\mathbb{E}_\omega$.

*Example 25.* Let $A : D(A) \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ be an unbounded linear operator. One can easily check that, $\psi(u,v) = \langle Au, Av\rangle$, for $u = (u_t)_{t\in\mathbb{N}}, v = (v_t)_{t\in\mathbb{N}} \in D(A)$ is a non-Archimedean bilinear form, which can be explicitly expressed by

$$\psi(u,v) = \sum_{t,s=0}^{\infty} \sigma_{ts}\, u_t \,.\, v_s$$

where $\sigma_{ts} = \langle Ae_t, Ae_s\rangle$, $t,s = 0,1,2,...$

Clearly, $\psi$ given above is well-defined. Indeed, for each $u = (u_t)_{t\in\mathbb{N}} \in D(A)$,

$$|u_t| \,.\, |\langle Ae_t, Ae_t\rangle|^{1/2} \leq |u_t|\, \|Ae_t\| \mapsto 0 \ \text{ as } \ t \mapsto \infty,$$

by $u = (u_t)_{t\in\mathbb{N}} \in D(A)$. Moreover, $D(A) \subset D(\psi)$.

We now consider an example of bilinear forms corresponding to diagonal operators. Let $(\lambda_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a sequence of terms such that $\lim_{t\to\infty} |\lambda_t| = \infty$. Define the operator $A$ by

$$\begin{cases} D(A) = \{u = (u_t)_{t\in\mathbb{N}} : \lim_{t\to\infty} |\lambda_t||u_t|\|e_t\| = 0\}, \\ \\ Au = \sum_{t=0}^{\infty} \lambda_t u_t e_t \ \text{ for each } \ u = \sum_{t=0}^{\infty} u_t e_t \in D(A). \end{cases}$$

One can easily check that $A$ is well-defined and is an unbounded linear operator on $\mathbb{E}_\omega$. Define $\psi(u,v) = \langle Au, v\rangle$ for each $u, v \in D(A)$ by

$$\psi(u,v) = \sum_{t=0}^{\infty} \omega_t \lambda_t u_t v_t \tag{5.2.5}$$

for all $u = \sum_{t=0}^{\infty} u_t e_t, \ v = \sum_{t=0}^{\infty} v_t e_t \in D(A)$.

Clearly, $\psi$ is well-defined. Indeed,

$$|\omega_t||\lambda_t||u_t||v_t| = (|\lambda_t||u_t|\|e_t\|)(|v_t|\|e_t\|)$$

for each $t \in \mathbb{N}$, and hence $\lim_{t\to\infty} |\omega_t||\lambda_t||u_t||v_t| = 0$, by $u \in D(A)$ and $v \in D(A) \subset \mathbb{E}_\omega$. Next, it is straightforward to check that $\psi$ is a non-Archimedean bilinear form.

If $\psi : D(\psi) \times D(\psi) \subset \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is a non-Archimedean bilinear form, then $q$ denotes the non-Archimedean *quadratic form* associated with $\psi$ and is defined by $q(u) = \psi(u,u)$ for each $u \in D(\psi) = D(q)$. Thus one can easily check that the relationship between the quadratic form $q$ and the non-Archimedean bilinear form $\psi$ can be expressed as

$$4 \,.\, \psi(u,v) = q(u+v) - q(u-v), \ \ \forall u, v \in D(\psi). \tag{5.2.6}$$

Furthermore,

$$2\ .\ [q(u)+q(v)] = q(u+v)+q(u-v) \tag{5.2.7}$$

for all $u,v \in D(\psi)$.

Let $\phi,\psi$ be $p$-adic bilinear forms. One defines the $p$-adic form sum of these non-Archimedean bilinear forms by setting

$$\begin{cases} D(\phi+\psi) = D(\phi)\cap D(\psi), \\ \\ (\phi+\psi)(u,v) = \phi(u,v)+\psi(u,v) \end{cases}$$

for all $u,v \in D(\phi)\cap D(\psi)$.

Similarly, $D(\lambda\ .\ \phi) = D(\phi)$ and $(\lambda\ .\ \phi)(u,v) = \lambda\ .\ \phi(u,v)$ for all $\lambda \in \mathbb{K}-\{0\}$ and $u,v \in D(\phi)$.

## 5.3 Closed and Closable non-Archimedean Bilinear Forms

Let $a$ be a non-Archimedean (unbounded) bilinear form. As in the classical context one defines the so-called $\phi$-convergence as follows.

**Definition 33.** A sequence $(x_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega$ is said to be $\phi$-convergent to $x \in \mathbb{E}_\omega$ if $(x_t)_{t\in\mathbb{N}} \in D(\phi)$, $x_t \mapsto x$ in $\mathbb{E}_\omega$ and $q(x_t - x_s) \mapsto 0$ in $\mathbb{K}$ as $t,s \mapsto \infty$.

The $\phi$-convergence of a sequence $(x_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega$ to $x \in \mathbb{E}_\omega$ will be denoted by $x = \phi - \lim_{t\to\infty} x_t$. As in the classical context, note that the $\phi$-limit, $x$, appearing in Definition 33, may or may not belong to $D(\phi)$.

*Example 26.* Consider the non-Archimedean bilinear form $\phi$ (respectively, its corresponding quadratic form $q$) given in Example 25 with $\mathbb{K} = \mathbb{Q}_p$, the field of $p$-adic numbers equipped with the $p$-adic absolute value $|\cdot|$. ($p \geq 2$ being a prime.) Suppose $\omega_t = p^t$ for each $t \in \mathbb{N}$ and that $\lambda_t = p^{-t}$ for each $t \in \mathbb{N}$.

Clearly, $\lim_{t\to\infty} |\lambda_t| = \infty$, and

$$\begin{cases} D(A) = \{u = (u_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{t\to\infty} p^{\frac{t}{2}}|u_t| = 0\}, \\ \\ Au = \sum_{t=0}^{\infty} p^t u_t e_t \ \text{ for each } \ u = \sum_{t=0}^{\infty} u_t e_t \in D(A), \ \text{ and} \end{cases}$$

$$q(u) = \sum_{t=0}^{\infty} u_t^2 \ \text{ for each } \ u = \sum_{t=0}^{\infty} u_t e_t \in D(A).$$

Here,

$$D(\phi) = D(q) = \{u = (u_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{t\to\infty} |u_t| = 0\}.$$

Clearly, $D(A) \subset D(\phi)$ since $|u_t| \leq p^{\frac{t}{2}}|u_t| \mapsto 0$ as $t \mapsto \infty$ whenever $u = (u_t)_{t\in\mathbb{N}} \in D(A)$.

Now consider the sequence defined by $y_t^s = p^{3t+s}$ for each $t, s \in \mathbb{N}$. One can easily check that $(y_s)_{s\in\mathbb{N}} \in D(A)$, $y_t^s \mapsto 0$ as $s \mapsto \infty$ for each $t \in \mathbb{N}$, and

$$q(y_s - y_r) = (p^s - p^r)^2 \sum_{t\in\mathbb{N}} p^{6t} \mapsto 0 \text{ in } \mathbb{Q}_p \text{ as } s, r \mapsto \infty$$

and hence $\phi - \lim_{s\to\infty} y_s = 0$.

**Definition 34.** A non-Archimedean bilinear form $\phi$ is said to be closed if $x = \phi - \lim_{t\to\infty} x_t$ yields $x \in D(\phi)$ and that $q(x_s - x) \mapsto 0$ as $s \mapsto \infty$.

**Theorem 15.** *Let $a : D(\phi) \times D(\phi) \mapsto \mathbb{K}$ be a bounded non-Archimedean bilinear form. Then $\phi$ is closed if and only if $D(\phi)$ is a closed subspace of $\mathbb{E}_\omega$.*

*Proof.* Suppose that $\phi$ is a closed bounded non-Archimedean bilinear form, i.e., $\phi$ is closed and there exists $M \geq 0$ such that

$$|\phi(u,v)| \leq M \,.\, \|u\| \,.\, \|v\|, \quad \forall (u,v) \in D(\phi) \times D(\phi).$$

Let $(u_s)_{s\in\mathbb{N}} \in D(\phi)$ such that $u_s \mapsto u$ in $\mathbb{E}_\omega$. So we have to show that $u \in D(\phi)$. Now from the boundedness of $\phi$ and the fact that $(u_s)_{s\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{E}_\omega$ ($u_s \mapsto u$ in $\mathbb{E}_\omega$) it is clear that $q(u_t - u_s) \mapsto 0$ as $t, s \mapsto \infty$ where $q$ is the quadratic form associated with $\phi$, and hence $u = \phi - \lim_{s\to\infty} u_s$. Using the closedness of $\phi$ it follows that $u \in D(\phi)$, and therefore $D(\phi)$ is a closed subspace of $\mathbb{E}_\omega$.

Conversely, suppose that $D(\phi)$ is a closed subspace of $\mathbb{E}_\omega$ and let $u = \phi - \lim_{s\to\infty} u_s$. Thus $(u_s)_{s\in\mathbb{N}} \in D(\phi)$, $u_s \mapsto u$ in $\mathbb{E}_\omega$ as $s \mapsto \infty$, and $q(u_t - u_s) \mapsto 0$ as $t, s \mapsto \infty$. Since $D(\phi)$ is closed, then $u \in D(\phi)$. Again, by the boundedness of $\phi$ and the fact $u_s \mapsto u$ in $\mathbb{E}_\omega$ as $s \mapsto \infty$ it follows that $q(u_t - u) \mapsto 0$ as $t \mapsto \infty$, and hence $\phi$ is closed.

### 5.3.1 Closedness of the Form Sum

**Definition 35.** Let $\phi_t$ ($t = 1, 2, \ldots n$) be $n$ non-Archimedean bilinear forms and let $q_t$ ($t = 1, \ldots n$) be the associated quadratic forms, respectively. One says that the sequence of non-Archimedean bilinear forms $(\phi_t)_{t=1,\ldots n}$ is *quadratically disjoint* if its satisfies

$$|q_1(u)| \neq |q_2(u)| \neq \ldots \neq |q_n(u)| \tag{5.3.1}$$

for each $0 \neq u \in D(\phi_1) \cap D(\phi_2) \cap \ldots \cap D(\phi_n)$.

**Theorem 16.** *Let $(\phi_t)_{t=1,\ldots n}$ be a (finite) sequence of bilinear forms such that each $\phi_t$ ($t = 1, 2, \ldots, n$) is closed. If (5.3.1) holds, then the form sum $\phi = \phi_1 + \phi_2 + \ldots. + \phi_n$ with domain $D(\phi) = D(\phi_1) \cap D(\phi_2) \cap \ldots \cap D(\phi_n)$ is closed.*

*Proof.* Suppose $\phi - \lim_{s\to\infty} u_s = u$ for some sequence $(u_s)_{s\in\mathbb{N}}$. Clearly, if the sequence $(u_s)_{s\in\mathbb{N}}$ is constant, then there is nothing to prove. Now suppose that $(u_s)_{s\in\mathbb{N}}$ is a nonconstant sequence. Consequently,

$$q_1(u_t - u_s) + q_2(u_t - u_s) + \ldots + q_n(u_t - u_s) \mapsto 0 \text{ as } t, s \mapsto \infty.$$

Clearly, among the natural integers, 1, 2, 3, ...,n, there exists $k_0$ such that

$$|q_1(u_t - u_s) + q_2(u_t - u_s) + \ldots + q_n(u_t - u_s)| = |q_{k_0}(u_t - u_s)|.$$

Namely,

$$|q_{k_0}(u_t - u_s)| = \max(|q_1(u_t - u_s)|, |q_2(u_t - u_s)|, \ldots, |q_n(u_t - u_s)|),$$

by using (5.3.1) and the fact $\mathbb{K}$ is non-Archimedean. And hence

$$|q_{k_0}(u_t - u_s)| \mapsto 0 \text{ as } t, s \mapsto \infty.$$

Consequently, $a_{k_0} - \lim_{s\to\infty} u_s = u$, and therefore $u \in D(\phi_{k_0})$ and $q_{k_0}(u_s - u) \mapsto 0$ as $s \mapsto \infty$. Using the fact that $|q_r(u_t - u_s)| < |q_{k_0}(u_t - u_s)|$ for each $r = 1, 2, \ldots n$ with $r \neq k_0$ it easily follows that $u \in D(\phi_r)$ and $q_r(u_s - u) \mapsto 0$ for each $k = 1, 2, \ldots, n$ with $r \neq k_0$ as $s \mapsto \infty$.

In summary, $u \in D(\phi)$ and $q(u_s - u) \mapsto 0$ as $s \mapsto \infty$, where $q = q_1 + q_2 + \ldots + q_n$, and therefore $\phi$ is closed.

*Remark 20.* (1) Theorem 6 still holds when the quadratically disjoint assumption, Eq. (5.3.1), is replaced by the *strictly increasing* assumption given by

$$|q_1(u)| < |q_2(u)| < \ldots < |q_n(u)|.$$

for each $0 \neq u \in D(\phi_1) \cap D(\phi_2) \cap \ldots \cap D(\phi_n)$.

(2) In the classical context one makes use of both the closedness and the positiveness of the real part of each form $\phi_t$ $(t = 1, 2, \ldots, n)$ to deduce the closedness of the form sum $\phi$. Here, the situation is much more tricky since there is no notion of positiveness in a general non-Archimedean field $\mathbb{K}$.

**Corollary 5.** *Let $\phi : D(\phi) \times D(\phi) \subset \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ be a closed non-Archimedean bilinear form. Suppose that*

$$\left|\sum_{t=0}^{\infty} \omega_t u_t^2\right| \neq |\phi(u,u)| \quad \textit{for all nonzero} \quad u = \sum_{t=0}^{\infty} u_t e_t \in D(\phi).$$

*Then the form sum $\psi$ defined by $\psi(u,v) := \langle u, v\rangle + \phi(u,v)$ for all $u, v \in D(\phi)$ is closed.*

*Proof.* Suppose that $\left|\sum_{t=0}^{\infty}\omega_t u_t^2\right| < |\phi(u,u)|$ each $u = \sum_{t=0}^{\infty} u_t e_t \in D(\phi)$ with $u \neq 0$. (when $\left|\sum_{t=0}^{\infty}\omega_t u_t^2\right| > |\phi(u,u)|$, one follows along the same lines as the case being treated.) Set $\xi(u,v) = \langle u,v\rangle$ for all $u,v \in \mathbb{E}_\omega$ where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{E}_\omega$ defined in (2.3.2). Clearly $\xi$ is bounded, by the Cauchy-Schwarz inequality. Now since $D(\xi) = \mathbb{E}_\omega$, then $\xi$ is closed, by Theorem 15. One completes the proof by setting $\psi = \phi + \xi$ and apply it to Theorem 6.

### 5.3.2 Construction of a non-Archimedean Hilbert Space Using Quadratic Forms

Let $\phi : D(\zeta) \times D(\zeta) \subset \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ be a non-degenerate non-Archimedean bilinear form. Define

$$\langle u,v\rangle_\zeta := \langle u,v\rangle + \zeta(u,v) \tag{5.3.2}$$

for all $u,v \in D(\zeta)$, and

$$\|u\|_q^2 := \max(|q(u)|, \|u\|^2) \tag{5.3.3}$$

for each $u \in D(\zeta)$, where $q$ is the quadratic form associated with the non-Archimedean bilinear form $\zeta$.

One requires the following assumption

$$|q(u+v)| \leq \max(|q(u)|, |q(v)|) \tag{5.3.4}$$

for all $u,v \in D(\zeta)$.

One can easily check that $\langle\cdot,\cdot\rangle_\zeta$ is symmetric, non-degenerate form on $D(\zeta)\times D(\zeta)$ with values in $\mathbb{K}$. Under assumption (5.3.4) it is clear that $\|\cdot\|_q$ is a non-Archimedean norm on $D(\zeta)$. Indeed, for all $u,v \in D(\zeta)$

$$\begin{aligned}
\|u+v\|_q^2 &= \max(|q(u+v)|, \|u+v\|^2) \\
&\leq \max\left[\max(|q(u)|, |q(v)|), \max(\|u\|^2, \|v\|^2)\right] \\
&= \max\left(|q(u)|, |q(v)|, \|u\|^2, \|v\|^2\right) \\
&\leq \max(\|u\|_q^2, \|v\|_q^2, \|u\|_q^2, \|v\|_q^2) \\
&= \max(\|u\|_q^2, \|v\|_q^2).
\end{aligned}$$

In other words, $\|u+v\|_q \leq \max(\|u\|_q, \|v\|_q)$, and hence $\|.\|_q$ is a non-Archimedean norm over $D(\zeta)$. Let $\mathbb{E}_q$ denote the normed non-Archimedean space $(D(\zeta), \|\cdot\|_q)$.

**Theorem 17.** *Let $\zeta$ be a non-degenerate non-Archimedean bilinear form over $D(\zeta)\times D(\zeta)$. Suppose that (5.3.4) holds. Then, $\zeta$ is closed if and only if $\mathbb{E}_q$ is complete.*

*Proof.* Suppose that $\zeta$ is closed and let $(u_s)_{s\in\mathbb{N}} \in D(\zeta)$ be a Cauchy sequence in $\mathbb{E}_q$. Thus $\|u_t - u_s\|_q^2 = \max(|q(u_t - u_s)|, \|u_t - u_s\|^2) \mapsto 0$ as $t, s \mapsto \infty$. Now from the fact that $\|\cdot\|_q^2 \geq |q(\cdot)|, \ \|\cdot\|^2$ it easily follows that

$$\|u_t - u_s\|, \ |q(u_t - u_s)| \mapsto 0 \ \text{ as } \ t, s \mapsto \infty,$$

and hence there exists $u \in \mathbb{E}_\omega$ such that $u_s \mapsto u$ as $s \mapsto \infty$ and $q(u_t - u_s) \mapsto 0$ in $\mathbb{K}$ as $t, s \mapsto \infty$. In other words, $(u_s)_{s\in\mathbb{N}}$ is $\zeta$-convergent to $u$. Since $\zeta$ is closed it follows that $u \in D(\zeta)$ and $q(u_s - u) \mapsto 0$ in $\mathbb{K}$ as $s \mapsto \infty$. In summary, $\|u_s - u\|_q \mapsto 0$ as $s \mapsto \infty$, and therefore, $\mathbb{E}_q$ is complete.

Conversely, suppose that $\mathbb{E}_q$ is complete and let $u = \zeta - \lim_{s\to\infty} u_s$ for some sequence $(u_s)_{s\in\mathbb{N}}$. Thus $(u_s)_{s\in\mathbb{N}} \in D(\zeta)$, $u_s \mapsto u$ in $\mathbb{E}_\omega$, and $q(u_t - u_s) \mapsto 0$ as $t, s \mapsto \infty$. It is routine to see that $\|u_t - u_s\|_q \mapsto 0$ as $t, s \mapsto \infty$. Now since $\mathbb{E}_q$ is complete there exists $v \in D(\zeta) = \mathbb{E}_q$ such that $\|u_s - v\|_q \mapsto 0$ as $s \mapsto \infty$. Now $\|u_s - v\|_q \geq \|u_s - v\|$, hence $u_s \mapsto v$ in $\mathbb{E}_\omega$ as $s \mapsto \infty$. Using the uniqueness of the limit we deduce that $u = v$. We then conclude that $q(u_s - u) \mapsto 0$ as $s \mapsto \infty$, that is, $q$ is closed.

*Remark 21.* In view of the proof of Theorem 17, if $\zeta$ is a non-degenerate form whose corresponding quadratic form satisfies (5.3.4), then $\zeta - \lim_{s\to\infty} u_s = u$ if and only if $\|u_t - u_s\|_q \mapsto 0$ as $t, s \mapsto \infty$. Furthermore, $u \in D(\zeta)$, $\zeta - \lim_{s\to\infty} u_s = u$ if and only if $\|u_s - u\|_q \mapsto 0$ as $s \mapsto \infty$.

If the non-degenerate form $\zeta$ is closed and if (5.3.4) holds, then the space $\mathbb{E}_q$ is called a $p$-adic Hilbert space when it is equipped with the $\mathbb{K}$-form given by (5.3.2). Furthermore, if one supposes that $\zeta$ is bounded with bound $\leq 1$ it is routine to check that the Cauchy-Schwarz inequality is satisfied. Indeed,

$$\begin{aligned} |\langle u, v\rangle_\zeta| &= |\langle u, v\rangle + \zeta(u, v)| \\ &\leq \max(|\langle u, v\rangle|, |\zeta(u, v)|) \\ &\leq \max(\|u\|\|v\|, \|u\|\|v\|) \\ &= \|u\|\|v\| \\ &\leq \|u\|_q\|v\|_q \ \text{ for all } \ u, v \in \mathbb{E}_q. \end{aligned}$$

### 5.3.3 Further Properties of the Closure

Let $\phi, \psi$ be non-Archimedean bilinear forms. One says that $\phi$ is an extension of $\psi$ and denote it $\psi \subset \phi$ if $D(\psi) \subset D(\phi)$ and $\psi(u, v) = \phi(u, v)$ for all $u, v \in D(\psi)$.

**Definition 36.** A non-Archimedean bilinear form $\phi$ is said to be *closable* if it has a closed extension.

**Theorem 18.** *Let $\phi$ be a non-degenerate non-Archimedean bilinear form. Suppose that $q$, the corresponding quadratic form to $\phi$ satisfies Eq. (5.3.4). If $\phi$ is closable, then $\phi - \lim_{s\to\infty} u_s = 0$ yields $q(u_s) \to 0$ as $s \mapsto \infty$.*

*Proof.* Let $\overline{\phi}$ denote a closed extension of $\phi$. Clearly, the statement $\phi - \lim_{s\to\infty} u_s = 0$ yields $\overline{\phi} - \lim_{s\to\infty} u_s = 0$ . Now if $\overline{q}$ is the quadratic form associated with $\overline{\phi}$ it is clear that $q(u) = \overline{q}(u)$ for each $u \in D(\phi) \subset D(\overline{\phi})$, and hence $q(u_s) = \overline{q}(u_s) = \overline{q}(u_s - 0) \mapsto 0$ as $s \mapsto \infty$, by the closedness of $\overline{\phi}$.

*Remark 22.* When $\phi$ is closable, the smallest closure in the sense of the extension of quadratic forms is called the *closure* of $\phi$ and denoted by $\overline{\phi}$. Clearly, $D(\phi) \subset D(\overline{\phi})$.

## 5.4 Representation of Bilinear Forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ by Linear Operators

**Definition 37.** *A bilinear form $\phi : D(\phi) \times D(\phi) \mapsto \mathbb{K}$ is said to be representable whether there exists a (possibly unbounded) linear operator $A : D(A) \mapsto \mathbb{E}_\omega$ such that*

$$\phi(u,v) = \langle Au, v\rangle, \quad \forall u \in D(A), v \in D(\phi).$$

In this section we examine the representation of (unbounded) bilinear forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ by linear operators. Namely, it will be shown that each non-degenerate bilinear form $\phi$ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ is representable whenever

$$\lim_{i\to\infty} \left( \frac{|\phi(e_i, e_j)|}{\|e_i\|} \right) = 0. \tag{5.4.1}$$

Moreover, if $A$ denotes the linear operator on $\mathbb{E}_\omega$ associated with the form $\phi$, then the adjoint $A^*$ of $A$ does exist provided that the following holds: for each $j \in \mathbb{N}$,

$$\lim_{i\to\infty} \left( \frac{|\phi(e_i, e_j)|}{\|e_i\|} \right) = \lim_{i\to\infty} \left( \frac{|\phi(e_j, e_i)|}{\|e_i\|} \right) = 0. \tag{5.4.2}$$

**Theorem 19.** *Let $\phi : D(\phi) \times D(\phi) \mapsto \mathbb{K}$ be a non-degenerate unbounded bilinear form. Then $\phi$ is representable whenever assumption (5.4.1) holds. Moreover, if $A$ denotes the linear operator associated with $\phi$, then the adjoint $A^*$ of $A$ exists whenever assumption (5.4.2) holds.*

*Proof.* Suppose that assumption (5.4.1) holds. For all $u = (u_i)_{i\in\mathbb{N}}, v = (v_j)_{j\in\mathbb{N}} \in D(\phi)$, write

$$\phi(u,v) = \sum_{i,j=0}^{\infty} \phi(e_i, e_j)\, u_i v_j, \ \text{ with } \ \forall j \in \mathbb{N}, \ \lim_{i\to\infty} \left( |u_i| \,.\, |\phi(e_i, e_j)|^{1/2} \right) = 0.$$

Define the linear operator $A$ on $\mathbb{E}_\omega$ as follows:

$$\begin{cases} D(A) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim_{i\to\infty} |u_i| \, \|Ae_i\| = 0\}, \\ \\ Au = \displaystyle\sum_{i,j\in\mathbb{N}} \left[ \frac{\phi(e_i, e_j)}{\omega_i} \right] (e'_j \otimes e_i) u, \quad \forall u = (u_i)_{i\in\mathbb{N}} \in D(A). \end{cases}$$

Obviously, $A$ is well-defined. Indeed, for all $j \in \mathbb{N}$,

$$\lim_{i\to\infty}\left|\frac{\phi(e_i,e_j)}{\omega_i}\right|\|e_i\| = \lim_{i\to\infty}\frac{|\phi(e_i,e_j)|}{\|e_i\|} = 0,$$

by using (5.4.1).

Now

$$Au = \sum_{j\in\mathbb{N}}\frac{1}{\omega_j}\left(\sum_{i\in\mathbb{N}} u_i\phi(e_i,e_j)\right)e_j, \quad \forall u=(u_i)_{i\in\mathbb{N}}\in D(A),$$

and hence for each $i \in \mathbb{N}$, $\langle Ae_i, e_i\rangle = \phi(e_i,e_i)$.

Moreover, $D(A) \subset D(\phi)$. Indeed, if $u = (u_i)_{i\in\mathbb{N}} \in D(A)$, then using the Cauchy-Schwarz inequality it follows that, $\forall i \in \mathbb{N}$,

$$\begin{aligned}
|u_i|^2\ |\phi(e_i,e_i)| &= |u_i|^2 . \|e_i\| . \left(\frac{|\phi(e_i,e_i)|}{\|e_i\|}\right)\\
&= |u_i|^2 . \|e_i\| . \left(\frac{|\langle Ae_i,e_i\rangle|}{\|e_i\|}\right)\\
&\le |u_i|^2 . \|e_i\| . \|Ae_i\|\\
&= (|u_i| . \|e_i\|) . (|u_i| . \|Ae_i\|),
\end{aligned}$$

and hence $\lim_{i\to\infty}\left(|u_i| . |\phi(e_i,e_i)|^{1/2}\right) = 0$, that is, $u \in D(\phi)$.

Note that $u_i v_k \phi(e_i,e_k) \to 0$ as $i,k \to \infty$, by using the fact that ($u \in D(A) \subset D(\phi)$ and $v \in D(\phi)$):

$$|u_i v_k \phi(e_i,e_k)| = \left(|u_i||\phi(e_i,e_k)|^{1/2}\right) . \left(|\phi(e_i,e_k)|^{1/2}|v_k|\right) \to 0, \text{ as } i,k\to\infty,$$

and hence

$$\sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{N}} u_i v_k \phi(e_i,e_k) = \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} u_i v_k \phi(e_i,e_k),$$

according to a result by Cassels [7]. Consequently, the following successive equalities are justified:

$$\begin{aligned}
\langle Au, v\rangle &= \sum_{k\in\mathbb{N}} \omega_k v_k \frac{1}{\omega_k}\left(\sum_{i\in\mathbb{N}} u_i\phi(e_i,e_k)\right)\\
&= \sum_{k\in\mathbb{N}} v_k\left(\sum_{i\in\mathbb{N}} u_i\phi(e_i,e_k)\right)\\
&= \sum_{i,k\in\mathbb{N}} \phi(e_i,e_k)u_i v_k\\
&= \phi(u,v)
\end{aligned}$$

for all $u = (u_i)_{i\in\mathbb{N}} \in D(A)$ and $v = (v_k)_{k\in\mathbb{N}} \in D(\phi)$.

Furthermore, the uniqueness of $A$ is guaranteed by the fact that $\phi$ is non-degenerate. It remains to show that $A^*$, the adjoint of $A$ exists; this can be done as in the bounded case.

*Example 27.* Consider the bilinear form defined by

$$\phi(u,v) = \sum_{i,j\in\mathbb{N}} \pi_{ij}\,.\,u_i v_j, \quad \forall u = (u_i)_{i\in\mathbb{N}}, v = (v_i)_{i\in\mathbb{N}} \in D(\phi)$$

where $(\pi_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{K}$ is an arbitrary sequence, and the domain $D(\phi)$ of $\phi$ is defined as follows:

$$D(\phi) = \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{i\to\infty}\left(|u_i|\,.\,|\pi_{ij}|^{1/2}\right) = 0\}.$$

Note that $\phi(e_i, e_j) = \pi_{ij}$ for all $i, j \in \mathbb{N}$ and hence an equivalent of assumption (5.4.1) is:

$$\lim_{i\to\infty} \frac{|\pi_{ij}|}{\|e_i\|} = 0. \tag{5.4.3}$$

Upon making assumption (5.4.3), the unique (possibly unbounded) linear operator associated with $\phi$ is given by

$$Au = \left(\sum_{i,j\in\mathbb{N}} \frac{\pi_{ij}}{\omega_i}\, e'_j \otimes e_i\right) u, \quad \forall u = (u_i)_{i\in\mathbb{N}} \in D(A)$$

where $D(A) = \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{i\to\infty} \|Ae_i\|\,.\,|u_i| = 0\}$.

If in addition, $\lim_{i\to\infty}\left(\frac{|\pi_{ij}|}{\|e_i\|}\right) = \lim_{i\to\infty}\left(\frac{|\pi_{ji}|}{\|e_i\|}\right) = 0$, then the adjoint $A^*$ of $A$ does exist.

**Corollary 6.** *Let $\phi : D(\phi)\times D(\phi) \mapsto \mathbb{K}$ be a non-degenerate (unbounded) symmetric bilinear form. Then $\phi$ is representable whenever assumption (5.4.1) holds. Moreover, if $A$ denotes the linear operator associated with $\phi$, then the adjoint $A^*$ of $A$ exists with $A = A^*$.*

## 5.5 Bibliographical Notes

This chapter consists of a preliminary work by the author on non-Archimedean bilinear forms. All the results of this chapter are quite new and can be found in Diagana **[11, 21]**.

*Chapter 6*

# Functions of Some Self-adjoint Linear Operators on $\mathbb{E}_\omega$

## 6.1 Introduction

Fractional powers of closed linear operators play a key role in many fields such as the theory of analytic semigroups, existence and uniqueness of solutions of solutions to some partial differential equations, stochastic processes, and enable us to handle the well-known *square root problem of Kato* (see, **[13]**, **[14]**, **[15]**, and **[16]**) [1].

Among other things, an important and challenging objective in a near future consists of formulating a non-Archimedean version of the square root problem of Kato. That obviously requires the introduction in the literature of a non-Archimedean theory of fractional powers for (general) closed unbounded linear operators.

This chapter considers some preliminary investigations by the author on integer powers as well as functions of (possibly unbounded) of some specific linear operators on $\mathbb{E}_\omega$. As an illustration, we will construct functions of a $2\times 2$ symmetric matrix on $\mathbb{Q}_p \times \mathbb{Q}_p$. However, one should point out that several questions related to fractional powers of general closed linear operators in the non-Archimedean context still remain.

## 6.2 Products and Sums of Diagonal Operators

Let $(\lambda_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ be a sequence satisfying (4.4.1) and let $A$ be a diagonal operator defined on $\mathbb{E}_\omega$ by

$$D(A) = \{x = (x_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_t |\lambda_t| . |x_t| . \|e_t\| = 0\},$$

and

$$Ax = \sum_{t\in\mathbb{N}} \lambda_t x_t e_t, \quad \forall x \in D(A).$$

Now, let $B \in U(\mathbb{E}_\omega)$ be another diagonal operator on $\mathbb{E}_\omega$ defined by,

[1] Note that the square root problem of Kato in the classical setting amounts to showing that for some regular linear operators $A$, $D(A^{1/2}) = D(A^{*1/2})$.

$$D(B) = \{x = (x_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_t |\mu_t| . |x_t| . \|e_t\| = 0\},$$

and

$$Bx = \sum_{t\in\mathbb{N}} \mu_t x_t e_t, \quad \forall x \in D(B),$$

where $(\mu_t)_{t\in\mathbb{N}} \subset \mathbb{K}$, then the algebraic sum $A+B$ of $A$ and $B$ is defined by

$$\begin{cases} D(A+B) = D(A)\cap D(B), \\ \\ (A+B)x = Ax + Bx, \end{cases}$$

for all $x \in D(A)\cap D(B)$.

As a straightforward consequence of Proposition 28 we have

**Corollary 7.** *Under (4.4.1), suppose that $|\mu_t| < |\lambda_t|$ for each $t \in \mathbb{N}$, then $A+B$ is self-adjoint. Furthermore,*

$$\rho(A+B) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_t + \mu_t, \ \forall t \in \mathbb{N}\}, \ \textit{and}$$

$$\|(A+B-\lambda)^{-1}\| = \sup_{t\in\mathbb{N}} \left(\frac{1}{|\lambda - (\lambda_t + \mu_t)|}\right)$$

*for each $\lambda \in \rho(A+B)$.*

*Proof.* First of all, note that $(A+B)x = \sum_{t\in\mathbb{N}} (\lambda_t + \mu_t) x_t e_t$ for each $x = (x_t)_{t\in\mathbb{N}} \in D(A+B)$, where

$$D(A+B) = \{x = (x_t)_{t\in\mathbb{N}} : \lim_{t\to\infty} |\lambda_t + \mu_t| . |x_t| . \|e_t\| = 0\}.$$

From $|\mu_t| < |\lambda_t|$ for each $t \in \mathbb{N}$ it follows that $|\lambda_t + \mu_t| = |\lambda_t|$ for all $t \in \mathbb{N}$, and hence, $D(A+B) = D(A)$.

Now, $A+B$ is well-defined since

$$\lim_{t\to\infty} |\lambda_t + \mu_t|\,|x_t|\,\|e_t\| = \lim_{t\to\infty} |\lambda_i|\ |x_t|\ \|e_t\| = 0.$$

Note that $A+B$ is diagonal with coefficients $\gamma_t = \lambda_t + \mu_t$, where $\lim_{t\to\infty} |\gamma_t| = \lim_{t\to\infty} |\lambda_t| = \infty$, by (4.4.1). To complete the proof one follows along the same line as in the proof of Proposition 28.

Let $(\mu_t) \subset \mathbb{K}$ be a sequence. Similarly, the product $AB$ of the diagonal operators $A$ and $B$ is defined by:

$$\begin{cases} D(AB) = \{x \in D(B) : Bx \in D(A)\}, \\ \\ (AB)x = A(Bx), \quad \forall x \in D(AB). \end{cases}$$

It can be easily checked that, $(AB)x = \sum_{t\in\mathbb{N}} \lambda_t \mu_t x_t e_t$, for each $x = (x_t)_{t\in\mathbb{N}} \in D(AB)$, where

$$D(AB) = \{x = (x_t)_{t\in\mathbb{N}} : \lim_{t\to\infty} |\lambda_t| . |\mu_t| . |x_t| . \|e_t\| = 0\}.$$

**Corollary 8.** *If* $\lim_{t\to\infty} |\lambda_t \mu_t| = \infty$*, then the product AB of A and B is self-adjoint. Furthermore,*

$$\rho(AB) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_t \mu_t,\ \forall t \in \mathbb{N}\}, \ \ \textit{and}$$

$$\|(AB-\lambda)^{-1}\| = \sup_{t\in\mathbb{N}} \left(\frac{1}{|\lambda_t\ \mu_t - \lambda|}\right)$$

*for each* $\lambda \in \rho(AB)$.

## 6.3 Integer Powers of Diagonal Operators

For $\mu = (\mu_t)_{t\in\mathbb{N}} \subset \mathbb{K}$, let $\mathbb{J}(\mu)$ denote the collection of $z \in \mathbb{Z}$ such that $\lim_{t\to\infty} |\mu_t{}^z| = \lim_{t\to\infty} |\mu_t|^z = \infty$, that is,

$$\mathbb{J}(\mu) = \{z \in \mathbb{Z} : \lim_{t\to\infty} |\mu_t{}^z| = \lim_{t\to\infty} |\mu_t|^z = \infty\}.$$

*Remark 23.* If $\lambda = (\lambda_t)_{t\in\mathbb{N}}, \mu = (\mu_t)_{t\in\mathbb{N}}$ are sequences of elements in $\mathbb{K}$, one has the following properties:

(1) If $z, z' \in \mathbb{J}(\lambda)$, then $z + z',\ zz' \in \mathbb{J}(\lambda)$;
(2) $\mathbb{J}(\lambda) = \mathbb{J}(|\lambda|)$;
(3) $\mathbb{J}(\lambda + \mu) = \mathbb{J}(\max(|\lambda|, \mu|))$ whenever $|\lambda| \neq |\mu|$;
(4) $\mathbb{J}(\mu) \subset \mathbb{J}(\lambda)$ whenever $|\mu| \leq |\lambda|$;
(5) $0 \notin \mathbb{J}(\lambda)$ for each $\lambda$;
(6) $\mathbb{J}(0) = \mathbb{Z}^- - \{0\}$, the set of nonzero negative integers;
(7) $\mathbb{J}(\lambda) \cap \mathbb{J}(\lambda^{-1}) = \{\emptyset\}$;
(8) $\mathbb{J}(1_{\mathbb{K}}) = \{\emptyset\}$.

**Definition 38.** Let $z \in \mathbb{J}(\lambda)$. Define integer powers $(A^z)_{z\in\mathbb{Z}-\{0\}}$ of the diagonal operator $A$ by:

$$\begin{cases} D(A^z) = \{x = (x_t)_{t\in\mathbb{N}} \subset \mathbb{K} :\ \lim_t |\lambda_t|^z \,.\, |x_t| \,.\, \|e_t\| = 0\}, \\ \\ A^z x = \sum_{t\in\mathbb{N}} \lambda_t{}^z x_t e_t \ \text{ for each } \ x = (x_t)_{t\in\mathbb{N}} \in D(A^z), \end{cases} \tag{6.3.1}$$

where $(\lambda_t)_{t\in\mathbb{N}} \subset \mathbb{K}$. Similarly, one defines $A^0 = I$, where $I$ is the identity operator on $\mathbb{E}_\omega$.

*Example 28.* Suppose $\mathbb{K} = \mathbb{Q}_p$ where $p \geq 2$ is a prime number. Consider the diagonal operator $A$ defined by:

$$D(A) = \{x = (x_t)_{t\in\mathbb{N}} \subset \mathbb{Q}_p :\ \lim_t |\lambda_t| \,.\, |x_t| \,.\, \|e_t\| = 0\},$$

and

$$Ax = \sum_{t\in\mathbb{N}} \lambda_t x_t e_t, \ \ \forall x \in D(A),$$

where $\lambda_t = p^{p^t}$ for each $t \in \mathbb{N}$.

It is easy to see that $\mathbb{J}(\lambda) = \mathbb{Z}^- - \{0\}$, the set of all nonzero negative integers. In this event, for each $z \in \mathbb{Z}^- - \{0\}$, one defines $A^z$ by:

$$D(A^z) = \{x = (x_t)_{t\in\mathbb{N}} \subset \mathbb{Q}_p : \lim_t p^{-zp^t}|x_i|\|e_t\| = 0\}$$

and

$$A^z x = \sum_{t\in\mathbb{N}} p^{zp^t} x_t e_t, \ \ \forall x \in D(A).$$

Using previous results, one can easily see that $A^z$ is self-adjoint and that

$$\rho(A^z) = \{\lambda \in \mathbb{Q}_p : \lambda \neq p^{p^t}, \ \ \forall t \in \mathbb{N}\}.$$

**Proposition 31.** *Let $z, z' \in \mathbb{J}(\lambda)$. If $A$ is a diagonal operator with $(\lambda_t)_{t\in\mathbb{N}} \subset \mathbb{K}$ as a corresponding sequence, then*

*(1) $A^z.A^{z'} = A^{z+z'}$;*
*(2) $(A^z)^{z'} = A^{zz'}$.*

*Proof.* (1) If $x = \sum_{t\in\mathbb{N}} x_t e_t \in D(A^z.A^{z'})$, then $A^z A^{z'} x = \sum_{t\in\mathbb{N}} {\lambda_t}^{z+z'} x_t e_t$. Next we use the fact that $z, z' \in \mathbb{J}(\lambda)$ yield $z + z' \in \mathbb{J}(\lambda)$ (see Remark 29).

(2) Similarly, if $x = \sum_{t\in\mathbb{N}} x_t e_t \in D((A^z)^{z'})$, then $(A^z)^{z'} x = \sum_{t\in\mathbb{N}} {\lambda_t}^{zz'} x_t e_t$. Using Remark 29 it easily follows that $(A^z)^{z'} = A^{zz'}$.

**Proposition 32.** *If $z \in \mathbb{J}(\lambda)$, then $A^z$ is self-adjoint. Furthermore, $\rho(A^z) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_t^z, \ \forall t \in \mathbb{N}\}$, and*

$$\|(A^z - \lambda)^{-1}\| = \sup_{t\in\mathbb{N}} \left( \frac{1}{|\lambda_t^z - \lambda|} \right)$$

*for each $\lambda \in \rho(A^z)$.*

*Proof.* First of all, note that $A^z x = \sum_{t\in\mathbb{N}} {\lambda_t}^z x_t e_t$, $\forall x = (x_t)_{t\in\mathbb{N}} \in D(A^z)$ with $D(A^z) = \{x = (x_t)_{t\in\mathbb{N}} \subset \mathbb{K} : \lim_{t\to\infty} |\lambda_t|^z \, |x_t| \, \|e_t\| = 0\}$. Since $z \in \mathbb{J}(\lambda)$ it follows that $A^z$ is well-defined.

Note that $A^z$ is a diagonal operator corresponding to $\gamma_t = {\lambda_t}^z$ with $\lim_{t\to\infty} |\gamma_t| = \infty$, by $z \in \mathbb{J}(\lambda)$. So to complete the proof one follows along the same line as in the proof of Proposition 28.

**Proposition 33.** *Let $A, B$ be diagonal operators on $\mathbb{E}_\omega$. If $\lambda = (\lambda_t)_{t\in\mathbb{N}}$, $\mu = (\mu_t)_{t\in\mathbb{N}}$ are respectively the corresponding sequences to the diagonal operators $A$ and $B$, and if $|\lambda_t| \neq |\mu_t|$ for each $t \in \mathbb{N}$ and $\mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+ \neq \emptyset$ ($\mathbb{Z}^+$ being the set of all natural numbers), then*

$$D((A+B)^z) = D(A^z) \cap D(B^z) = D((A+B)^{*z}),$$

*for each $z \in \mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+$.*

*Proof.* Using the fact that $|\lambda_t| \neq |\mu_t|$ for each $t \in \mathbb{N}$ it easily follows that

$$|\lambda_t + \mu_t|^z = \max(|\lambda_t|^z, |\mu_t|^z) \tag{6.3.2}$$

for all $t \in \mathbb{N}$, $z \in \mathbb{Z}^+$. In particular, (6.3.2) holds for each $z \in \mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+$.

Now the operator $(A+B)^z$ is defined by:

$$(A+B)^z x = \sum_{t \in \mathbb{N}} (\lambda_t + \mu_t)^z x_t e_t$$

for each $x = (x_t)_{t \in \mathbb{N}} \in D((A+B)^z)$, where

$$\begin{aligned} D((A+B)^z) &= \{x = (x_t)_{t \in \mathbb{N}} \subset \mathbb{K} : \lim_{t \to \infty} |\lambda_t + \mu_t|^z \, |x_t| \, \|e_t\| = 0\} \\ &= \{x = (x_t)_{t \in \mathbb{N}} : \lim_{t \to \infty} \max(|\lambda_t|^z, |\mu_t|^z) \, |x_t| \, \|e_t\| = 0\} \\ &= D(A^z) \cap D(B^z). \end{aligned}$$

It is also clear that $(A+B)^z$ is self-adjoint for each $z \in \mathbb{J}(\lambda) \cap \mathbb{J}(\mu) \cap \mathbb{Z}^+$, and hence $(A+B)^z = (A+B)^{*z}$.

*Remark 24.* In view of Definition 38, the point now is how to define integer powers of a general unbounded linear operator $A$ defined by

$$A\psi = \sum_{ts} a_{ts} (e'_s \otimes e_t) \psi, \;\; \forall \psi \in D(A).$$

It is clear that whether $A$ is a self-adjoint linear operator on $\mathbb{E}_\omega$ whose point spectrum is given by a sequence $(\gamma_t)_{t \in \mathbb{N}}$ and if the sequence $(f_t)_{t \in \mathbb{N}}$ is its corresponding orthogonal eigenfunctions, then $A$ can be seen as a diagonal operator in $(f_t)_{t \in \mathbb{N}}$, and hence Definition 38 can be applied to it. More generally, if $f : \mathbb{K} \mapsto \mathbb{K}$ is an analytic functions whose domain $Dom(f)$ contains the sequence $(\gamma_t)_{t \in \mathbb{N}}$, then one can define $f(A)$. The latter point will be investigated in Section 6.4.

Another interesting question concerns *fractional powers* of a general unbounded linear operator $A$ defined by

$$A\phi = \sum_{ts} a_{ts} (e'_s \otimes e_t) \phi, \;\; \forall \phi \in D(A),$$

including diagonal operators. A partial answer to this question is given as follows: consider $\mathcal{M}$, the set of all (complete) non-Archimedean fields $(\mathbb{K}, |\cdot|)$ such that if $\lambda \in \mathbb{K}$, then $\lambda^q \in \mathbb{K}$ for some $q \in \mathbb{Q}$, and $|\lambda^q| = |\lambda|^q$. Clearly, if $\mathbb{K} \subset \mathcal{M}$, then the fractional powers of every diagonal operator on (as a non-Archimedean Hilbert space over $\mathbb{K}$) $\mathbb{E}_\omega$ are defined as for diagonal operators, see Definition 38. Next, one applies the previous method to those self-adjoint operators on $\mathbb{E}_\omega$ whose point spectrums are given by sequences as above to define their functions.

## 6.4 Functions of Self-Adjoint Operators

This section is a generalization of the previous one and introduces functions of some specific self-adjoint operators on $\mathbb{E}_\omega$. Namely, we construct functions of a self-adjoint (possibly unbounded) operator $T$ on $\mathbb{E}_\omega$ whose point spectrum $\sigma_p(T)$, is given by a sequence $(\gamma_t)_{t\in\mathbf{N}} \subset \mathbb{K}$. For that, let $(h_t)_{t\in\mathbb{N}}$ denote the sequence of eigenfunctions associated with the eigenvalues $(\gamma_t)_{t\in\mathbf{N}}$, that is,

$$Th_t = \gamma_t h_t$$

for each $t \in \mathbb{N}$. As in the classical context, one can easily check that $\langle h_t, h_s\rangle = \varpi_t \delta_{ts}$ for some $(\varpi_t)_{t\in\mathbb{N}} \subset \mathbb{K} - \{0\}$. For the sake of simplicity, we suppose that $(h_t)_{t\in\mathbb{N}}$ is an orthogonal base for $\mathbb{E}_\omega$. To achieve the latter goal, we assume that there exists a nontrivial isometric linear bijection $V$ such that:

$$Ve_t = h_t \text{ for all } t \in \mathbb{N}.$$

Consequently,

$$\langle h_t, h_s\rangle = \langle Ve_t, Ve_s\rangle = \varpi_t \delta_{ts},$$

and

$$\|h_t\|^2 = |\varpi_t| = |\omega_t|$$

for each $t \in \mathbb{N}$.

Clearly, for each $u \in \mathbb{E}_\omega$, $u = \sum_{t\in\mathbb{N}} u_t h_t$ with $\lim_{t\to\infty} |u_t| \|h_t\| = 0$, where $\|h_t\| = |\omega_i|^{1/2}$, for each $t \in \mathbb{N}$. Moreover, the operator $T$ can be rewritten in the base $(h_t)_{i\in\mathbb{N}}$ as

$$\begin{cases} D(T) = \{u = (u_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim_{t\to\infty} |u_t||\gamma_t|\,\|h_t\| = 0\}, \\ \\ Tu := \sum_{t\in\mathbb{N}} \gamma_t u_t h_t, \ \text{ for each } \ u = (u_t)_{t\in\mathbb{N}} \in D(T). \end{cases}$$

Thus if $f : \mathbb{K} \mapsto \mathbb{K}$ is analytic, and if each $\gamma_t$ for $t \in \mathbb{N}$ belongs to $Dom(f)$, the domain of $f$, one then defines $f(T)$ by

$$\begin{cases} D(f(T)) = \{u = (u_t)_{t\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim_{t\to\infty} |u_t||f(\gamma_t)|\,\|h_t\| = 0\}, \\ \\ f(T)u := \sum_{t\in\mathbb{N}} f(\gamma_t) u_t h_t, \ \text{ for each } \ u = (u_t)_{t\in\mathbb{N}} \in D(f(T)). \end{cases}$$

Using Proposition 28, the following holds:

**Proposition 34.** *Under previous assumptions, the operator $f(T)$ defined above is self-adjoint. Furthermore,*

$$\rho(f(T)) = \{\gamma \in \mathbb{K} : \lambda \neq f(\gamma_t), \ \forall t \in \mathbb{N}\}, \quad \textit{and}$$

$$\|(f(T) - \lambda)^{-1}\| = \sup_{t\in\mathbb{N}} \left( \frac{1}{|f(\gamma_t) - \lambda|} \right)$$

*for each $\lambda \in \rho(f(T))$.*

If $f, g : \mathbb{K} \mapsto \mathbb{K}$ are analytic functions such that $\sigma_p(T) \subset Dom(f) \cap Dom(g)$, then the following hold:

(1) $(\alpha f)(T) = \alpha f(T)$ for all $\alpha \in \mathbb{K}$;
(2) $D(f(T) + g(T)) \subset D((f+g)(T))$;
(3) $D(f(T)g(T)) \subset D((fg)(T))$;
(4) If $f(x) = a_t x^t + a_{t-1} x^{t-1} + ... + a_1 x + a_0 \in \mathbb{K}[x]$, then

$$f(T) = a_t T^t + a_{t-1} T^{t-1} + ... + a_1 T + a_0 I$$

with $D(f(T)) = D(T^t)$;
(5) $\|f(T)\| = \sup_{t \in \mathbb{N}} |f(\gamma_t)|$.

## 6.5 Functions of Some Symmetric Square Matrices Over $\mathbb{Q}_p \times \mathbb{Q}_p$

Throughout of this section, we let $p$ denote an *odd prime*. This section illustrates in some extent our previous construction. Namely, if $f : \mathbb{Q}_p \mapsto \mathbb{Q}_p$ is an analytic function, we give an explicit formula for $f(T)$ where $T$ is a $2\times 2$ symmetric matrix over $\mathbb{Q}_p \times \mathbb{Q}_p$ and $\sigma_p(T) \subset \mathbb{Q}_p$. In particular, if $f$ is either $f(x) = x^n$ or $f(x) = x^{-n}$, or $f(x) = \exp_p(x)$ for $x \in \mathbb{Z}_p$ such that $|x| < p^{\frac{-p}{p-1}}$, then the formula for $f(T)$ will be considered.

Here, we consider $2\times 2$ (symmetric) square matrices $T$ over $\mathbb{Q}_p \times \mathbb{Q}_p$:

$$T = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{Q}_p.$$

From now on, we suppose that $a, b \in \mathbb{Q}_p - \{0\}$. Moreover, the direct sum $\mathbb{Q}_p \times \mathbb{Q}_p$ will be equipped with the non-Archimedean norm and the inner product defined respectively by: For all $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Q}_p \times \mathbb{Q}_p$,

$$\| \begin{pmatrix} x \\ y \end{pmatrix} \| = \max(|x|, |y|)$$

and

$$\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = xu + yv$$

where $|\cdot|$ is the $p$-adic absolute value.

In view of the above, $\|e_1\| = \| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \| = 1$, $\|e_2\| = \| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \| = 1$, and

$$\langle e_s, e_t \rangle = \delta_{st}, \quad s, t = 1, 2,$$

where $\delta_{st}$ are the Kronecker symbols.

It is not hard to see that $\sigma(T) = \sigma_P(T) = \{a-b, a+b\}$. Moreover, it is clear that $\phi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are respectively the eigenvectors associated with the eigenvalues $a-b$ and $a+b$.

Consider the nontrivial linear bijection $V$ defined on $\mathbb{Q}_p \times \mathbb{Q}_p$ by

$$V = \begin{pmatrix} 1 & 1 \\ & \\ -1 & 1 \end{pmatrix}$$

whose inverse $V^{-1}$ is given by

$$V^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ & \\ 1/2 & 1/2 \end{pmatrix}.$$

Moreover, $Ve_1 = \phi$ and $Ve_2 = \psi$. Clearly, $\{\phi, \psi\}$ is also an orthogonal base of $\mathbb{Q}_p \times \mathbb{Q}_p$. Furthermore, each $\begin{pmatrix} x \\ y \end{pmatrix} = xe_1 + ye_2 \in \mathbb{Q}_p \times \mathbb{Q}_p$ can be uniquely expressed as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left[\frac{x-y}{2}\right].\phi + \left[\frac{x+y}{2}\right].\psi.$$

Clearly, $T$ considered in $\{\phi, \psi\}$ can be rewritten as

$$T = \begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}, \quad a, b \in \mathbb{Q}_p,$$

or

$$T\begin{pmatrix} x \\ y \end{pmatrix} = (a-b)\left[\frac{x-y}{2}\right].\phi + (a+b)\left[\frac{x+y}{2}\right].\psi.$$

Using our previous setting, if $f : \mathbb{Q}_p \mapsto \mathbb{Q}_p$ is an analytic function whose domain contains $\sigma(T)$, then

$$\begin{aligned} f(T)\begin{pmatrix} x \\ y \end{pmatrix} &= \left[\frac{x-y}{2}\right] f(a-b)\phi + \left[\frac{x+y}{2}\right] f(a+b)\psi \\ &= \left[\frac{x-y}{2}\right] f(a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \left[\frac{x+y}{2}\right] f(a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \left[\frac{x-y}{2}\right] f(a-b) + \left[\frac{x+y}{2}\right] f(a+b) \\ -\left[\frac{x-y}{2}\right] f(a-b) + \left[\frac{x+y}{2}\right] f(a+b) \end{pmatrix} \end{aligned}$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Q}_p \times \mathbb{Q}_p$.

Clearly,

$$f(T) = \begin{pmatrix} \dfrac{f(a-b)+f(a+b)}{2} & \dfrac{f(a+b)-f(a-b)}{2} \\ \dfrac{f(a+b)-f(a-b)}{2} & \dfrac{f(a-b)+f(a+b)}{2} \end{pmatrix}.$$

Note that $f(T)$ is also a symmetric matrix whose eigenvalues are given by:

$$\sigma(f(T)) = \{f(a-b), f(a+b)\} = f(\sigma(T)).$$

### 6.5.1 The Powers of the Matrix $T$

We are interested to finding explicit expressions of both $T^n$ and $T^{-n}$ for $n = 1,2,3,\ldots$. For that we take respectively $f(x) = x^n$, $x \in \mathbb{Q}_p$ and $f(x) = x^{-n}$, $x \in \mathbb{Q}_p - \{0\}$.

We have

$$T^n = \begin{pmatrix} \dfrac{(a-b)^n+(a+b)^n}{2} & \dfrac{(a+b)^n-(a-b)^n}{2} \\ \dfrac{(a+b)^n-(a-b)^n}{2} & \dfrac{(a-b)^n+(a+b)^n}{2} \end{pmatrix}.$$

If $a-b, a+b \in \mathbb{Q}_p - \{0\}$, then

$$T^{-n} = \begin{pmatrix} \dfrac{(a-b)^{-n}+(a+b)^{-n}}{2} & \dfrac{(a+b)^{-n}-(a-b)^{-n}}{2} \\ \dfrac{(a+b)^{-n}-(a-b)^{-n}}{2} & \dfrac{(a-b)^{-n}+(a+b)^{-n}}{2} \end{pmatrix}.$$

### 6.5.2 Exponential of the Matrix $T$

To define $e^T$, the exponential matrix of $T$, we suppose that both $a-b$ and $a+b$ belong to the domain of the $p$-adic exponential, i.e., $a+b, a-b \in \mathbb{Z}_p$ such that $|a-b|,\ |a+b| < p^{\frac{-p}{p-1}}$.

Under previous assumptions, $e^T$ is defined by:

$$e^T = \begin{pmatrix} \dfrac{\exp_p(a-b)+\exp_p(a+b)}{2} & \dfrac{\exp_p(a+b)-\exp_p(a-b)}{2} \\ \dfrac{\exp_p(a+b)-\exp_p(a-b)}{2} & \dfrac{\exp_p(a-b)+\exp_p(a+b)}{2} \end{pmatrix},$$

where $\exp_p(\cdot)$ denotes the $p$-adic exponential function.

As previously, $e^T$ is also symmetric with eigenvalues given by:

$$\sigma(e^T) = \{\exp_p(a-b), \exp_p(a+b)\} = \exp_p(\sigma(T)).$$

## 6.6 Open Problems

*Problem 1.* Let $T$ be an arbitrary (possibly unbounded) self-adjoint linear operator on $\mathbb{E}_\omega$. Construct fractional powers of $T$.

*Problem 2.* Let $T$ be an arbitrary (possibly unbounded) normal linear operator on $\mathbb{E}_\omega$. As in *Problem 1*, Can we construct fractional powers of $T$?

*Problem 3.* Formulate a non-Archimedean version of the Kato's square root problem for normal operators. For more on the classical version of the square root problem of Kato for (unbounded) normal operators, we refer the reader to Diagana**[19]**.

*Problem 4.* Let $A, B$ be (unbounded) self-adjoint operators on $\mathbb{E}_\omega$ such that $A - B \in B_2(\mathbb{E}_\omega)$. Suppose that both $f(A)$ and $f(B)$ exist, where $f : \mathbb{K} \mapsto \mathbb{K}$ is an analytic function. Find (necessary) sufficient conditions under which $f(A) - f(B) \in B_2(\mathbb{E}_\omega)$. In this event, approximate $Q(f(A) - f(B))$ in terms of $Q(A - B)$ ($Q(T)$ being the Hilbert-Schmidt norm of $T \in B_2(\mathbb{E}_\omega)$).

*Problem 5.* Let $A, B$ be (unbounded) self-adjoint operators on $\mathbb{E}_\omega$ such that $A - B \in B_1(\mathbb{E}_\omega)$. Suppose that both $f(A)$ and $f(B)$ exist for some analytic function $f : \mathbb{K} \mapsto \mathbb{K}$. Find sufficient conditions under which $f(A) - f(B) \in B_1(\mathbb{E}_\omega)$. In this event, find a relationship between $\mathrm{tr}(f(A) - f(B))$ in terms of $\mathrm{tr}(A - B)$ ($\mathrm{tr}(T)$ being the trace of $T \in B_1(\mathbb{E}_\omega)$).

## 6.7 Bibliographical Notes

The results of this chapter are quite new and entirely based upon results in Diagana**[22]**.

*Chapter 7*

# One-Parameter Family of Bounded Linear Operators on Free Banach Spaces

## 7.1 Introduction

Let $(\mathbb{K},+,\cdot,|\cdot|)$ be a (complete) non-Archimedean valued field and let $\Omega_r$ be the closed ball of $\mathbb{K}$ centered at 0 with radius $r$, that is,

$$\Omega_r = \{\kappa \in \mathbb{K} : |\kappa| \leq r\}.$$

As we have previously seen, $\Omega_r$ is also open in $\mathbb{K}$. Furthermore, every ball $\Omega_r$ is an additive subgroup of $\mathbb{K}$. From now on, the radius $r$ of the ball $\Omega_r$ will be suitably chosen so that the series, which defines the $p$-adic exponential converges. Indeed, let $\mathbb{K} = \mathbb{Q}_p$ be the field of $p$-adic numbers ($p \geq 2$ being a prime) equipped with the usual $p$-adic valuation $|\cdot|$ and let $\Omega_r = \{q \in \mathbb{Q}_p : |q| \leq r\}$. As we have seen in (1.3.4) of Subsection 1.1.3., the $p$-adic exponential

$$exp_p(x) := \sum_{t \geq 0} \frac{x^t}{t!}$$

is not always well-defined and analytic for each $x \in \mathbb{Q}_p$. However, it does converge for all $x \in \mathbb{Z}_p$ such that $|x| < r = p^{-\frac{1}{p-1}}$. For more on these we refer the reader to Subsection 1.1.3 or **[1, 40, 66]**.

In this chapter we provide the reader with a brief conceptualization of a non-Archimedean counterpart of the classical $C_0$-*semigroups* in connection with the formalism of linear operators on free Banach and non-Archimedean Hilbert spaces.

The present chapter is mainly motivated by the solvability of $p$-adic differential and partial differential equations, as strong (mild) solutions to the Cauchy problem related to several classes of differential and partial differential equations arising in the classical context can be explicitly expressed through $C_0$-semigroups, see, e.g., **[57]** and **[58]**.

As for the $p$-adic exponential defined above, here, the parameter of a given $C_0$-semigroup belongs to one of those clopen balls $\Omega_r$ whose radius $r$ will be suitably chosen. Let us mention however that the idea of considering one-parameter families of bounded linear operators on balls such as $\Omega_r$ was first initiated in **[1]** for bounded symmetric operators on $\mathbb{Q}_p$. Here, it goes back to consider those issues within the framework of free Banach and non-Archimedean Hilbert spaces while a development of a non-Archimedean operator theory is underway.

One of the consequences of the ongoing discussion is that if $\mathbb{K} = \mathbb{Q}_p$ and if $A$ is a bounded linear operator on a free Banach space $\mathbb{E}$ satisfying $\|A\| \leq r$ with $r = p^{-\frac{1}{p-1}}$, then the function defined by

$$v(x) = \left(\sum_{t\geq 0} \frac{(xA)^t}{t!}\right) u_0, \quad x \in \Omega_r$$

for a fixed $u_0 \in \mathbb{E}$, is the solution to the homogeneous $p$-adic differential equation given by

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t), \quad t \in \Omega_r \\ \qquad u(0) = u_0 \end{cases}$$

## 7.2 Basic Definitions

Let $(\mathbb{E}, \|\cdot\|)$ be a free Banach space. Throughout the rest of this chapter, we consider families $(T(\kappa))_{\kappa\in\Omega_r} : \mathbb{E} \mapsto \mathbb{E}$ of bounded linear operators. We always suppose that $r$ is suitably chosen so that $\kappa \mapsto T(\kappa)$ is well-defined.

**Definition 39.** Let $r > 0$ be a real number. A family $(T(\kappa))_{\kappa\in\Omega_r} : \mathbb{E} \mapsto \mathbb{E}$ of bounded linear operators will be called a *semigroup* of bounded linear operators on $\mathbb{E}$ if

(1) $T(0) = I_{\mathbb{E}}$, where $I_{\mathbb{E}}$ is the unit operator of $\mathbb{E}$;
(2) $T(\kappa+\kappa') = T(\kappa)T(\kappa')$ for all $\kappa, \kappa' \in \Omega_r$.

The semigroup $(T(\kappa))_{\kappa\in\Omega_r}$ will be called of class $C_0$ or *strongly continuous* if the following additional condition holds

(3) $\lim_{\kappa\to 0} \|T(\kappa)x - x\| = 0$ for each $x \in \mathbb{E}$.

*Remark 25.* One should point out that a semigroup $(T(\kappa))_{\kappa\in\Omega_r}$ of bounded linear operators is not only a semigroup but also a group. Every $T(\kappa)$ is invertible, the inverse being $T(-\kappa)$, according to (1)-(2) of Definition 39. Moreover, it is an infinite abelian group. Thus throughout the rest of this chapter the expression "group" (respectively, $C_0$-group) will be preferred to that of "semigroup" (respectively, $C_0$-semigroup).

*Remark 26.* A group $(T(\kappa))_{\kappa\in\Omega_r}$ will be called *uniformly continuous* if the following additional condition holds

(4) $\lim_{\kappa\to 0} \|T(\kappa) - I_{\mathbb{E}}\| = 0$.

**Definition 40.** If $(T(\kappa))_{\kappa\in\Omega_r}$ is a group as above, then the linear operator $A$ defined by

$$\begin{cases} D(A) = \{x \in \mathbb{E} : \lim_{\kappa\to 0}\left(\dfrac{T(\kappa)x - x}{\kappa}\right) \text{ exists}\}, \\ \\ Ax = \lim_{\kappa\to 0}\left(\dfrac{T(\kappa)x - x}{\kappa}\right), \text{ for each } x \in D(A) \end{cases}$$

is called the infinitesimal generator associated with the group $(T(\kappa))_{\kappa\in\Omega_r}$.

*Remark 27.* (1) Note that if $(T(\kappa))_{\kappa\in\Omega_r}$ is a group on $\mathbb{E}$ and if $(e_t)_{t\in\mathbb{N}}$ denotes the orthogonal basis for $\mathbb{E}$, then $T(\kappa)$ for each $\kappa \in \Omega_r$ can be expressed as, $\forall x = \sum_{t\in\mathbb{N}} x_t e_t \in \mathbb{E}$, by

$$T(\kappa)x = \sum_{t\in\mathbb{N}} x_t T(\kappa)e_t,$$

where

$$\forall s \in \mathbb{N}, \; T(\kappa)e_s = \sum_{t\in\mathbb{N}} a_{ts}(\kappa)e_t \text{ with } \lim_{t\to\infty} |a_{ts}(\kappa)|\|e_t\| = 0.$$

(2) Using (1) one can easily see that for each $0 \neq \kappa \in \Omega_r$,

$$\forall s \in \mathbb{N}, \; \left(\frac{T(\kappa) - I_{\mathbb{E}}}{\kappa}\right) e_s = \left(\frac{a_{ss}(\kappa) - 1}{\kappa}\right) e_s + \sum_{t\neq s} \frac{a_{ts}(\kappa)}{\kappa} e_t$$

with $\lim_{t\neq s, t\to\infty} |a_{ts}(\kappa)|\|e_t\| = 0.$

(3) If $(T(\kappa))_{\kappa\in\Omega_r}$ is a group on $\mathbb{E}$, then its infinitesimal generator $A$ may or may not be a bounded linear operator on $\mathbb{E}$.

## 7.3 Properties of non-Archimedean $C_0$-Groups

In this chapter we mainly focus on general groups, and strongly continuous groups of bounded linear operators on general free Banach spaces. We begin with the following example:

*Example 29.* Take $\mathbb{K} = \mathbb{Q}_p$ the field of $p$-adic numbers. Consider the ball $\Omega_r$ of $\mathbb{Q}_p$ with $r = p^{-\frac{1}{p-1}}$. Let $(\mathbb{X}, \|\cdot\|)$ be a free Banach space over $\mathbb{Q}_p$ and let $(f_t)_{t\in\mathbb{N}}$ be its canonical orthogonal base. Define for each $q \in \Omega_r$ and for $x = \sum_{t\geq 0} x_t f_t \in \mathbb{X}$ the family of linear operators

$$T(q)x = \sum_{t\geq 0} x_t e^{\mu_t q} f_t$$

where $(\mu_t)_{t\in\mathbb{N}} \subset \Omega_r$ is a sequence of nonzero elements.

It is routine to check that the family $(T(q))_{q\in\Omega_r}$ is well-defined.

**Proposition 35.** *The family* $(T(q))_{q\in\Omega_r}$ *of linear operators given above is a* $C_0$*-group of bounded linear operators whose infinitesimal generator is the (bounded) diagonal operator* $A$ *defined by*

$$Ax = \sum_{t\geq 0} \mu_t x_t f_t \ \ \textit{for each} \ \ x = \sum_{t\geq 0} x_t f_t \in \mathbb{X}.$$

*Proof.* First, note that $T(q)$ is analytic on the ball $\Omega_r$. It is routine to check that $(T(q))_{q\in\Omega_r}$ is a family of bounded linear operators on $\mathbb{X}$. Indeed, for each $q\in\Omega_r$,

$$T(q)f_t = e^{\mu_t q}\ f_t = \left(\sum_{s\geq 0}\frac{\mu_t^s q^s}{s!}\right) f_t, \ \ \forall t\in\mathbb{N},$$

and hence

$$||T(q)|| = \left|\left(\sum_{s\geq 0}\frac{\mu_t^s q^s}{s!}\right)\right| < \infty,$$

by the fact that $q\mu_t\in\Omega_r$ for each $t\in\mathbb{N}$.

Furthermore, one can easily check that

(1) $T(0) = I_{\mathbb{X}}$;
(2) $T(q+q') = T(q)T(q')$ for all $q,q'\in\Omega_r$;
(3) $\lim_{q\to 0}\|T(q)x - x\| = 0$ for each $x\in\mathbb{X}$.

Thus $(T(q))_{q\in\Omega_r}$ is a $C_0$-group of bounded linear operators.

Now let $B$ be the infinitesimal generator of $(T(q))_{q\in\Omega_r}$. It remains to show that $A = B$. First of all, let us show that $D(B) = \mathbb{X}$ $(= D(A))$. Clearly, for each $0\neq q\in\Omega_r$,

$$\frac{T(q)f_t - f_t}{q} = \left(\frac{e^{\mu_t q}-1}{q}\right) f_t$$

for each $t\in\mathbb{N}$, and hence

$$D(B) = \{x = (x_t)_{t\in\mathbb{N}} : \lim_{t\to\infty}|x_t|\ .\ \left\|\frac{T(q)f_t - f_t}{q}\right\| = 0\} = \mathbb{X},$$

by

$$|x_t|\ .\ \left\|\frac{T(q)f_t - f_t}{q}\right\| \leq \frac{(|x_t|\|f_t\|)}{|q|} \mapsto 0 \ \text{ as } \ t\mapsto\infty,$$

for each $x = \sum_{t\in\mathbb{N}} x_t f_t \in \mathbb{X}$.

To complete the proof it suffices to prove that

$$\left\|Af_t - \left(\frac{T(q)f_t - f_t}{q}\right)\right\| \mapsto 0 \ \text{ as } \ q\mapsto 0.$$

The latter is actually obvious since $\left(\dfrac{e^{\mu_t q}-1}{q}\right)\mapsto\mu_t$ as $q\mapsto 0$, and hence $B = A$ is the infinitesimal generator of the $C_0$-group $(T(q))_{q\in\Omega_r}$.

In the next theorem, we take $\mathbb{K} = \mathbb{Q}_p$ where $p \geq 2$ is prime. Note also that it is a natural generalization of Example 29.

**Theorem 20.** *Let $A$ be a bounded linear operator on $\mathbb{X}$ such that $\|A\| < r = p^{-\frac{1}{p-1}}$. Then $A$ is the infinitesimal generator of an uniformly continuous group of bounded operators $(T(q))_{q \in \Omega_r}$.*

*Proof.* Suppose that $A$ is a bounded linear operator on $\mathbb{X}$ with $\|A\| < r = p^{-\frac{1}{p-1}}$ and set, for each $q \in \Omega_r$,

$$T(q) = \sum_{s \geq 0} \frac{(qA)^s}{s!}. \tag{7.3.1}$$

Clearly, the series given by (7.3.1) converges in norm and defines a family of bounded linear operators on $\mathbb{E}$, by $|q| \,.\, \|A\| < r$. It is also routine to check that $T(0) = I_{\mathbb{X}}$, $T(q+q') = T(q)T(q')$ for all $q, q' \in \Omega_r$.

It remains to show that $(T(q))_{q \in \Omega_r}$ given above is uniformly continuous. Indeed, $0 \neq q \in \Omega_r$, one has

$$T(q) - I_{\mathbb{X}} = qA \left\{ \sum_{s \geq 0} \frac{(qA)^s}{(s+1)!} \right\},$$

and hence

$$\left\| \frac{T(q) - I_{\mathbb{X}}}{q} - A \right\| \leq \|A\| \,.\, \|T(q) - I_{\mathbb{X}}\| < \|T(q) - I_{\mathbb{X}}\|. \tag{7.3.2}$$

Now, $\|T(q) - I_{\mathbb{X}}\| \leq |q| \,.\, \|A\| \,.\, \|\zeta(q)\|$, where $\zeta(q) = \sum_{s \geq 0} \frac{(qA)^s}{(s+1)!}$ converges, and hence

$$\lim_{q \mapsto 0} \|T(q) - I_{\mathbb{X}}\| = 0. \tag{7.3.3}$$

Consequently,

$$\lim_{q \mapsto 0} \left\| \frac{T(q) - I_{\mathbb{X}}}{q} - A \right\| = 0,$$

by using both (7.3.2) and (7.3.3).

*Remark 28.* (i) Note that the mapping $\Omega_r \mapsto B(\mathbb{E}_\omega)$, $q \mapsto T(q)$ is analytic. Furthermore,

$$\frac{dT(q)}{dt} = AT(q) = T(q)A.$$

(ii) An abstract version of Theorem 20 , i.e., in a general non-Archimedean valued field $\mathbb{K}$, remains an open problem.

Now let $(\mathbb{K}, |\cdot|)$ be a (complete) non-Archimedean valued field and let $\Omega_r \subset \mathbb{K}$ be a clopen, were $r$ is chosen so that $\Omega_r \mapsto B(\mathbb{X})$, $\kappa \mapsto T(\kappa)$ is well-defined.

**Theorem 21.** *Let $(T(\kappa))_{\kappa\in\Omega_r}$ be a $C_0$-group and let $A$ be its infinitesimal generator. Suppose that there exists $M > 0$ such that $\|T(\kappa)\| \leq M$ for each $\kappa \in \Omega_r \subset \mathbb{K}$. Then, for each $x \in D(A)$, $T(\kappa)x \in D(A)$ for each $\kappa \in \Omega_r$. Furthermore,*

$$\left(\frac{dT(\kappa)}{d\kappa}\right)x = AT(\kappa)x = T(\kappa)Ax.$$

*Proof.* The proof, in some extent, is similar to that of the classical one, however for the sake of clarity, we will provide the reader with all details. Let $x \in D(A)$ and let $0 \neq \kappa \in \Omega_r$. Using Definition 39, Definition 40, and the boundedness of the $C_0$-group $T(\kappa)$, it easily follows that

$$\left(\frac{T(\kappa)-I_{\mathbb{X}}}{\kappa}\right)T(\kappa')x = T(\kappa')\left(\frac{T(\kappa)-I_{\mathbb{X}}}{\kappa}\right)x \tag{7.3.4}$$

$$\mapsto T(\kappa')Ax \tag{7.3.5}$$

as $\kappa \mapsto 0$.

Consequently, $T(\kappa')x \in D(A)$ and $AT(\kappa')x = T(\kappa')Ax$, by (7.3.4). Furthermore, since

$$T(\kappa')\left(\frac{T(\kappa)-I_{\mathbb{X}}}{\kappa}\right)x \mapsto T(\kappa')Ax, \text{ as } \kappa \mapsto 0$$

it follows that the right derivative of $T(\kappa')x$ is $T(\kappa')Ax$.

To complete the proof, we have to show that for each $0 \neq \kappa' \in \Omega_r$, the left derivative of $T(\kappa')x$ exists and is $T(\kappa')Ax$. Note that if $\sigma, \sigma' \in \Omega_r$, then so is $\sigma - \sigma'$, by using

$$|\sigma - \sigma'| \leq \max(|\sigma|, |\sigma'|) < r.$$

Now

$$\lim_{\kappa\to 0}\left(\frac{T(\kappa')x - T(\kappa'-\kappa)x}{\kappa} - T(\kappa')x\right) =$$

$$\lim_{\kappa\to 0} T(\kappa'-\kappa)\left(\frac{T(\kappa)x - x}{\kappa} - Ax\right) + \lim_{\kappa\to 0}\left[T(\kappa'-\kappa)Ax - T(\kappa')Ax\right].$$

Clearly,

$$\lim_{\kappa\to 0} T(\kappa'-\kappa)\left(\frac{T(\kappa)x - x}{\kappa} - Ax\right) = 0,$$

by $\|T(\sigma)\| \leq M$ for each $\sigma \in \Omega_r$.

Using the strong continuity of the group $T(\kappa)$ it follows that

$$\lim_{\kappa\to 0}\left[T(\kappa'-\kappa)Ax - T(\kappa')Ax\right] = 0.$$

Consequently,

$$\lim_{\kappa\to 0}\left(\frac{T(\kappa')x - T(\kappa'-\kappa)x}{\kappa} - T(\kappa')x\right) = 0,$$

and hence the left derivative of $T(\kappa')x$ exists and equals $T(\kappa')Ax$. This completes the proof.

*Remark 29.* One of consequences of Theorem 21 is that the function $v(t) = T(t)u_0$, $t \in \Omega_r$ for some $u_0 \in D(A)$, is the solution to the homogeneous $p$-adic differential equation given by

$$\begin{cases} \dfrac{du}{dt} = Au(t), \;\; t \in \Omega_r, \\ \\ u(0) = u_0, \end{cases} \tag{7.3.6}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is the infinitesimal generator of the $C_0$-group $(T(t)_{t\in\Omega_r}$ and $u : \Omega_r \mapsto D(A)$ is a $\mathbb{X}$-valued function.

## 7.4 Existence of Solutions to Some p-adic Differential Equations

The present section continues the above-mentioned discussion. Namely, we study the existence and uniqueness of a solution to the Cauchy problem for the homogeneous $p$-adic differential equation

$$\begin{cases} \dfrac{du}{dt} = Au(t), \;\; t \in \mathbb{Z}_p, \\ \\ u(0) = u_0 \in \mathbb{Q}_p, \end{cases} \tag{7.4.1}$$

where $A$ is a self-adjoint linear operator on $C(\mathbb{Z}_p, \mathbb{Q}_p)$ whose spectrum $\sigma(A) = \sigma_p(A) = \{\lambda_k\}_{k\in\mathbb{N}}$ with $\sigma_p(A)$ being the point spectrum of $A$. For that, the first task consists of constructing the spectral decomposition of $A$. Then utilizing the spectral decomposition of $A$ and upon making some additional assumptions on $\sigma(A)$, it will be shown that (7.4.1) has a unique solution, which can be explicitly expressed as

$$u(t) = \sum_{i\in\mathbb{N}} e^{t\lambda_i}\mathbb{E}_i(u_0), \;\; \forall t \in \mathbb{U}_p,$$

where $(\mathbb{E}_i)_{i\in\mathbb{N}}$ is called spectral projection associated with $A$ in the sense that $\mathbb{E}_i\mathbb{E}_j = \mathbb{E}_j\mathbb{E}_i = \delta_{ij}\mathbb{E}_i$ for all $i, j \in \mathbb{N}$, and $\mathbb{U}_p = \{u \in \mathbb{Z}_p : |u| < p^{-\frac{1}{p-1}}\}$. However, one should point out that the question, which consists of the existence and uniqueness of a solution to the corresponding inhomogeneous equation to (7.4.1) is open. Furthermore, except results of [31, 32] on the existence of solutions to (7.4.1) in the case when $A$ is closed in some non-Archimedean Banach space, it is quite unclear whether our results can be compared with the rest of the existing literature on $p$-adic differential equations.

To discuss the existence of solutions to (7.4.1), we first establish a spectral decomposition for $A$, which enables us to construct the $C_0$-semigroup associated $A$. To do so, we require the following assumptions:

(H.1) The operator $A$ is a self-adjoint on $C(\mathbb{Z}_p, \mathbb{Q}_p)$ with spectrum $\sigma(A)$ consisting of its eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Each eigenvalue is of (finite) multiplicity $m_l$ that is equal to the dimension of the corresponding eigenspace.

(H.2) There exists an orthonormal base $(f_k^l)_{k\in\mathbb{N}}$ of eigenvectors for $C(\mathbb{Z}_p, \mathbb{Q}_p)$.

(H.3) $\sigma(A) \subset \mathbb{U}_p$ where $\mathbb{U}_p = \{u \in \mathbb{Z}_p : |u| < p^{-\frac{1}{p-1}}\}$.

**Theorem 22.** *Under assumptions (H.1)-(H.2), for each $u \in D(A)$,*

$$Au = \sum_{k\in\mathbb{N}} \lambda_k \sum_{l=1}^{m_k} \langle u, f_k^l\rangle f_k^l = \sum_{k\in\mathbb{N}} \lambda_k \mathbb{E}_k(u)$$

*where $\mathbb{E}_k(u) = \sum_{l=1}^{m_k} \langle u, f_k^l\rangle f_k^l$ for each $k \in \mathbb{N}$, is called spectral projection associated with $A$ in the sense that $\mathbb{E}_i\mathbb{E}_j = \mathbb{E}_j\mathbb{E}_i = \delta_{ij}\mathbb{E}_i$ for all $i, j \in \mathbb{N}$.*

*Proof.* Using assumption (H.2) it easily follows that each $u \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ can be expressed in the orthonormal base $(f_k^l)_{k\in\mathbb{N}}$ as follows:

$$u = \sum_{k\in\mathbb{N}} \sum_{l=1}^{m_k} \langle u, f_k^l\rangle f_k^l = \sum_{k\in\mathbb{N}} \mathbb{E}_k(u) \ \text{ with } \ \lim_{k\to\infty} \left\| \sum_{l=1}^{m_k} \langle u, f_k^l\rangle f_k^l \right\|_\infty = 0,$$

where $\mathbb{E}_k(u) = \sum_{l=1}^{m_k} \langle u, f_k^l\rangle f_k^l$.

Now

$$\begin{aligned} Au &= \sum_{k\in\mathbb{N}} \sum_{l=1}^{m_k} \langle u, f_k^l\rangle A f_k^l \\ &= \sum_{k\in\mathbb{N}} \lambda_k \sum_{l=1}^{m_k} \langle u, f_k^l\rangle f_k^l \\ &= \sum_{k\in\mathbb{N}} \lambda_k \mathbb{E}_k(u) \end{aligned}$$

provided that $\lim_{k\to\infty} \left\| \lambda_k \sum_{l=1}^{m_k} \langle u, f_k^l\rangle f_k^l \right\|_\infty = \lim_{k\to\infty} \|\lambda_k \mathbb{E}_k(u)\|_\infty = 0$, that is, $u \in D(A)$.

Clearly, for each $u \in C(\mathbb{Z}_p, \mathbb{Q}_p)$,

$$\begin{aligned} \mathbb{E}_i\mathbb{E}_j(u) &= \sum_{l=1}^{m_i} \langle \mathbb{E}_j(u), f_i^l\rangle f_i^l \\ &= \sum_{l=1}^{m_i} \langle \sum_{r=1}^{m_j} \langle u, f_j^r\rangle f_j^r, f_i^l\rangle f_i^l \\ &= \sum_{l=1}^{m_i} \sum_{r=1}^{m_j} \langle u, f_j^r\rangle \langle f_j^r, f_i^l\rangle f_i^l, \end{aligned}$$

and hence $\mathbb{E}_i\mathbb{E}_j(u) = \mathbb{E}_j\mathbb{E}_i(u) = \delta_{ij}\mathbb{E}_i(u)$ for all $i, j \in \mathbb{N}$.

Now if $f : \mathbb{Z}_p \cap \sigma(A) \mapsto \mathbb{Q}_p$ is analytic, one defines $f(A)$ as follows:

$$\begin{cases} D(f(A)) := \{u \in C(\mathbb{Z}_p, \mathbb{Q}_p) : \lim_{i\to\infty} |f(\lambda_i)| . \|\mathbb{E}_i(u)\|_\infty = 0\}, \\ \\ f(A)u = \sum_{i\in\mathbb{N}} f(\lambda_i)\mathbb{E}_i(u), \quad \forall u \in D(f(A)). \end{cases}$$

In particular, under assumption (H.3) and letting $f(t) = e^t$ for each $t \in \mathbb{U}_p$ it follows that for each $t \in \mathbb{U}_p$,

$$e^{tA}u = \sum_{i\in\mathbb{N}} e^{\lambda_i t}\mathbb{E}_i(u), \;\; u \in C(\mathbb{Z}_p, \mathbb{Q}_p).$$

It is then easy to see that $(e^{tA})_{t\in\mathbb{U}_p}$ is an analytic $C_0$-group, which turns out to be an infinite group satisfying $e^{tA} = I$ for $t = 0$ with $I$ being the identity operator of $C(\mathbb{Z}_p, \mathbb{Q}_p)$. Moreover, using the fact that $A\mathbb{E}_i(u_0) = \lambda_i\mathbb{E}_i(u_0)$ for each $i \in \mathbb{N}$, one can easily see that

$$u(t) = \sum_{i\in\mathbb{N}} e^{\lambda_i t}\mathbb{E}_i(u_0), \;\; t \in \mathbb{U}_p$$

is the only solution to (7.4.1).

The previous discussion can be formulated as follows:

**Theorem 23.** *Under assumptions (H.1)-(H.2)-(H.3), the first-order differential equation, (7.4.1), has a unique solution, which can be explicitly expressed as*

$$u(t) = \sum_{i\in\mathbb{N}} e^{\lambda_i t}\mathbb{E}_i(u_0), \;\; t \in \mathbb{U}_p,$$

where $\mathbb{E}_i(u_0) = \sum_{l=1}^{m_i} \langle u_0, f_i^l \rangle f_i^l$ for $i \in \mathbb{N}$.

*Remark 30.* For $r > 0$, let $\mathbb{U}_r = \{u \in \mathbb{Q}_p : |u| \leq r\}$. Using Theorem 23, one easily sees that the Cauchy problem for the p-adic partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Au(t,x), \;\; t \in \mathbb{Z}_p, \; x \in \mathbb{U}_r \\ \\ u(0,x) = u_0(x), \;\; x \in \mathbb{U}_r \end{cases} \tag{7.4.2}$$

has also a unique solution, $u : \mathbb{Z}_p \times \mathbb{U}_r \mapsto \mathbb{Q}_p$, which can be explicitly expressed as

$$u(t,x) = \sum_{i\in\mathbb{N}} e^{\lambda_i t}\mathbb{E}_i(u_0(x)), \;\; t \in \mathbb{U}_p,$$

where $\mathbb{E}_i(u_0(x)) = \sum_{l=1}^{m_i} \langle u_0(x), f_i^l \rangle f_i^l$ for $i \in \mathbb{N}$.

## 7.5 Open Problems

*Problem 1.* Consider the nonhomogeneous $p$-adic differential equation

$$\begin{cases} \dfrac{du}{dt} = Au(t) + f(t), \;\; t \in \Omega_r, \\ \\ u(0) = u_0 \in \mathbb{Q}_p, \end{cases} \tag{7.5.1}$$

where $f : \Omega_r \mapsto \mathbb{Q}_p$ is a continuous function.

Express the solution to this equation in terms of $u_0,\ f$, and the $C_0$-group associated the multiplication operator

$$Au = Q(t)u, \;\; \forall u \in C(\mathbb{Z}_p, \mathbb{Q}_p),$$

where $Q = \sum_{s=0}^{\infty} q_s f_s \in C(\mathbb{Z}_p, \mathbb{Q}_p)$.

*Problem 2.* Prove the existence of solutions to the first-order $p$-adic differential equation

$$\begin{cases} \dfrac{du}{dt} = Au(t) + f(t), \;\; t \in \mathbb{Z}_p, \\ \\ u(0) = u_0 \in \mathbb{Q}_p, \end{cases} \tag{7.5.2}$$

and the first-order $p$-adic partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Au(t,x) + f(t,x), \;\; t \in \mathbb{Z}_p, x \in \Omega_r \\ \\ u(0,x) = u_0(x) \in C(\Omega_r, \mathbb{Q}_p), \end{cases} \tag{7.5.3}$$

where $A$ is a self-adjoint linear operator on $C(\mathbb{Z}_p, \mathbb{Q}_p)$ whose spectrum $\sigma(A) = \sigma_p(A) = \{\lambda_k\}_{k \in \mathbb{N}}$ with $\sigma_p(A)$ being the point spectrum of $A$ and $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ (respectively $f : \mathbb{Z}_p \times \Omega_r \mapsto \mathbb{Q}_p$ is jointly continuous).

Express the solution to this equation in terms of $u_0,\ f$, and the $C_0$-group associated the operator $A$.

## 7.6 Bibliographical Notes

The results of this chapter are quite new and entirely based upon results in Diagana **[23]**.

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